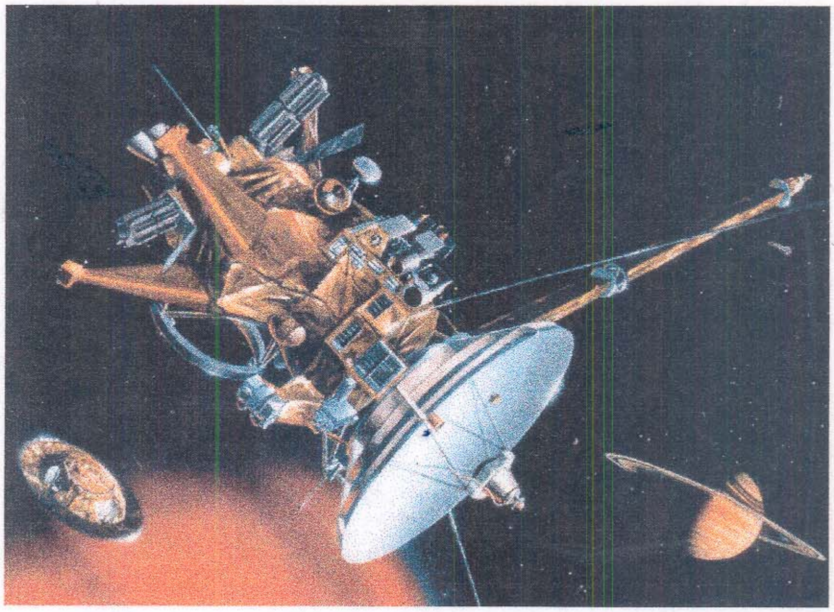
MAE/CE 362
- DYNAMICS -



CLASS NOTES
Version 2.0

by
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FALL 2000

MAE/CE 362 - Dynamics

Fall 2000

1999-01 Catalog Data:	MAE/CE 362: Dynamics 3 Hrs. Kinematics and kinetics of particles and systems of particles with applications to central force motion, impact, relative motion, vibrations, and variable mass systems. Dynamics of rigid bodies in plane motion, relative motion in rotating coordinates, and gyroscopic motion. Prerequisite: MAE/CE 271.
Textbook:	"Vector Mechanics for Engineers - Statics and Dynamics," 6th. Edition by Beer and Johnston, McGraw Hill, New York, NY, 1997. ISBN 0-07-005365-0.
Section/Time/Room:	Section No. 01; MW 5:30 p.m. - 6:50 p.m.; TH N158 Section No. 02; TR 8:00 a.m. - 9:20 a.m.; TH N152
Instructor:	Dr. Lin (Section No. 01); Dr. Gilbert (Section No. 02)
Office:	Dr. Lin: Technology Hall N268; Dr. Gilbert: Optics 301E
Phone:	Dr. Lin: (256) 824-6325; Dr. Gilbert: (256) 824-6029; Department (Ms. Beth Floyd): (256) 824-6154
E-Mail:	Lin: lin@eb.uah.edu; Gilbert: jag@eb.uah.edu
Office Hours:	To be announced.
Prerequisites:	Force and moment concepts; Newton's laws and basic concepts of physical laws; calculus and integration.
Grading:	3 In-Class Exams = 60% Homework = 10% Final Exam = 30%

Assignments will usually be due the first class meeting of each week. No late homework will be taken into consideration.

MAE/CE 362 - Dynamics

COURSE GUIDELINES

The aim of Mechanical/Civil Engineering 362 (Dynamics) is to explain and demonstrate the application of principles for the solution of problems in particle and rigid body dynamics. It is important to recognize that this subject requires a great deal of practice.

While typical students are aware of the forces encountered in statics, they do not exhibit the same familiarity with the motion that occurs in dynamics. This often requires a psychological adjustment in the attitude of students from the physical cause to the geometrical effect, (as well as a serious effort to avoid confusion between them). Students who are inclined to use their intuition in statics are often surprised to learn that it may fail to provide the correct solution to a problem in dynamics. Logic is the key to success in this discipline. Students are trained to think their way through a technical situation in a systematic manner, and that is a good exercise for anyone who wants to be a good engineer.

Keep in mind that students are required to be present from the beginning to the end of each semester, attend all classes, and take all examinations according to their assigned schedule. In case of absence, students are expected to satisfy the instructor that the absence was for good reason. For excessive cutting of classes, or for dropping the course without following the official procedure, students may fail the course. Use of a solutions manual or "old problem sets" is discouraged. Unauthorized use will constitute an act of academic misconduct.

Homework assignments must be done on only one side of 8 1/2" x 11" paper. Each page must contain the following (in the upper right hand corner): Your name, the date, and page __ of __. All final answers must be boxed and converted (SI to US or US to SI). Homework is due at the beginning of the class on the date prescribed. Work must be legible and should be done in pencil. Problems shall be restated prior to solution. Free-body diagrams (FBDs) shall be drawn for problems requiring such. Loose sheets shall be stapled together in the upper left hand corner.

MAE/CE 362 - Dynamics

COURSE SCHEDULE FOR MW CLASS

Textbook: "Vector Mechanics for Engineers - Statics and Dynamics," 6th. Edition by Beer and Johnston, McGraw Hill, New York, NY, 1997. ISBN 0-07-005365-0.

Prerequisite: MAE/CE 271.

DATE	CHAP./SECT.: TOPICS	HOMEWORK[1]
	KINEMATICS OF PARTICLES	
8- 23	introduction, distribution of course syllabus, discussion of course requirements	
28	11/1-3: rectilinear motion	11:6,14,17
30	11/4-6: uniformly accelerated motion, relative motion, curvilinear motion	11:33,35,54
9- 6	11/9-12: curvilinear motion, projectile motion	11:99,103,120
11	11/13,14: tangential & normal components, polar coordinates	11:139,144,158
13	catch up, review, and problem solving (See review and summary pages 659-662.)	
18	exam number 1	
	KINETICS OF PARTICLES: NEWTON'S SECOND LAW	
20	12/1-6: Newton's second law, equations of motion	12:6,19,36,44
25	12/7-10: angular momentum, Newton's law of gravitation	12:68,80,86
27	catch up, review, and problem solving (See review and summary pages 720-723.)	
	KINETICS OF PARTICLES: WORK AND ENERGY METHODS	
10- 2	13/1-5: work & energy	13:10,22,41
4	13/6-9: potential energy, conservation of energy	13:55,62,70,96
9	13/10,11: impulse and momentum	13:124,132,151
11	catch up, review, and problem solving (See review and summary pages 816-821.)	
16	exam number 2	
18	13/12-15: impact	13:158,174

[1] Assignments represent minimum requirements and may be supplemented by the instructor.

DATE	CHAP./SECT.: TOPICS	HOMEWORK
	SYSTEMS OF PARTICLES	
23	14/1-6: systems of particles	14:8,9,25
25	14/7,8: systems of particles	14:35,45,55
	KINEMATICS OF RIGID BODIES	
30	15/1-4: translation and rotation, general plane motion	15:6,10,21
11- 1	15/3,4: general plane motion	15:39,52
6	15/7,8,10,11: instantaneous center, absolute & relative acceleration, Coriolis acceleration	15:55,80,90,113,133
8	catch up, review, and problem solving (See review and summary pages 876-879 and 979-985.)	
13	exam number 3	
	KINETICS OF RIGID BODIES IN PLANE MOTION	
15	16/1-4,6,7: equations of motion of rigid bodies	16:7,17,29,34
20	16/8;17/1-7: constrained plane motion, work & energy	16:84,137
27	catch up, review, and problem solving (See review and summary pages 1039-1040.)	
29	17/8-10: impulse & momentum, course evaluation	17:12,38,62
12- 4	catch up, review, and problem solving (See review and summary on pages 1039-1040 and 1098-1101.)	

FINAL EXAM: Tuesday, December 12, 2000 from 11:30-2:00 p.m in TH S104.[2]

Note: The above schedule is intended as a guide. Exam dates and homework assignments are subject to change at the discretion of the instructor.

[2] The exam will be cumulative with problems from Chapters 16 and 17.

MAE/CE 362 - Dynamics

COURSE SCHEDULE FOR TR CLASS

Textbook: "Vector Mechanics for Engineers - Statics and Dynamics," 6th. Edition by Beer and Johnston, McGraw Hill, New York, NY, 1997. ISBN 0-07-005365-0.

Prerequisite: MAE/CE 271.

DATE	CHAP./SECT.: TOPICS	HOMEWORK[3]
	KINEMATICS OF PARTICLES	
8- 24	introduction, distribution of course syllabus, discussion of course requirements	
29	11/1-3: rectilinear motion	11:6,14,17
31	11/4-6: uniformly accelerated motion, relative motion, curvilinear motion	11:33,35,54
9- 5	11/9-12: curvilinear motion, projectile motion	11:99,103,120
7	11/13,14: tangential & normal components, polar coordinates	11:139,144,158
12	catch up, review, and problem solving (See review and summary pages 659-662.)	
14	exam number 1	
	KINETICS OF PARTICLES: NEWTON'S SECOND LAW	
19	12/1-6: Newton's second law, equations of motion	12:6,19,36,44
21	12/7-10: angular momentum, Newton's law of gravitation	12:68,80,86
26	catch up, review, and problem solving (See review and summary pages 720-723.)	
	KINETICS OF PARTICLES: WORK AND ENERGY METHODS	
28	13/1-5: work & energy	13:10,22,41
10- 3	13/6-9: potential energy, conservation of energy	13:55,62,70,96
10	13/10,11: impulse and momentum	13:124,132,151
12	catch up, review, and problem solving (See review and summary pages 816-821.)	
17	exam number 2	
19	13/12-15: impact	13:158,174

[3]Assignments represent minimum requirements and may be supplemented by the instructor.

DATE	CHAP./SECT.: TOPICS	HOMEWORK
	SYSTEMS OF PARTICLES	
24	14/1-6: systems of particles	14:8,9,25
26	14/7,8: systems of particles	14:35,45,55
	KINEMATICS OF RIGID BODIES	
31	15/1-4: translation and rotation, general plane motion	15:6,10,21
11- 2	15/3,4: general plane motion	15:39,52
7	15/7,8,10,11: instantaneous center, absolute & relative acceleration, Coriolis acceleration	15:55,80,90,113,133
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16	16/1-4,6,7: equations of motion of rigid bodies	16:7,17,29,34
21	16/8;17/1-7: constrained plane motion, work & energy	16:84,137
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FINAL EXAM: Tuesday, December 12, 2000 from 11:30-2:00 p.m. in TH S104[4]

Note: The above schedule is intended as a guide. Exam dates and homework assignments are subject to change at the discretion of the instructor.

[4] The exam will be cumulative with problems from Chapters 16 and 17.

CHAPTER 11 - KINEMATICS OF PARTICLES

11.1 Introduction

Statics: The analysis of bodies which are at rest or moving with a constant velocity. Makes use of $\Sigma\,\underline{F} = 0$ and $\Sigma\,\underline{M} = 0$ (equilibrium equations).

Dynamics: The analysis of bodies in motion. Makes use of $\Sigma\,\underline{F} = m\,\underline{a}$ and $\Sigma\,\underline{M} = I\,\underline{\alpha}$ (equations of motion).

Kinematics: Studies the geometry of motion but does not consider what causes it. This represents the right hand side of the governing equations.

Kinetics: The study of the relationship between motion and what causes it. The entire governing equations are taken into consideration.

11.2 Rectilinear Motion Of Particles

Definition: *Rectilinear motion* occurs when a particle moves in a straight line.

In the following sections involving rectilinear motion, displacement, velocity, and acceleration are often specified as a scalar quantity associated with a plus or minus sign. However, these quantities are really vectors where the scalar (magnitude) is associated with a unit vector (direction).

Definition: The location of a point measured along an axis relative to a fixed origin is called the *position coordinate*.

The motion of a particle is known when the position coordinate is specified as a function of time as

$$\boxed{x = x(t) = f(t)} \qquad \textbf{(11.2-1)}$$

where the function is either given graphically or expressed as a parametrical relation.

Coordinates are expressed in feet (U.S.) and in meters (MKS/SI) while time is expressed in seconds in both systems. The conversion between the units of distance is 1 ft = 0.3048 m. Other conversions can be found adjacent to the inside back cover of the text.

Example: Referring to Figure 1, specify the position vectors of the points labeled as P_1 and P_2. The dimensions are given in meters.

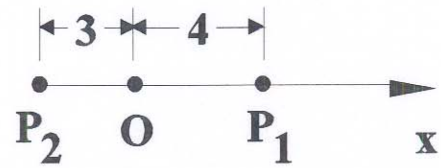


Figure 1. Two particles on a line.

The first step is to specify the origin for the coordinate system at O. Then the coordinate axis is labeled with the variable "x" and the positive coordinate direction denoted using an arrowhead.

Definition: Position Vector

In the scalar approach, the *position vector* measured with respect to the origin is simply denoted by the position coordinate of the point and a "+" or "-" sign. For the particles depicted in Figure 1,

$$x_1 = 4\ m \qquad and \qquad x_2 = -3\ m \quad . \tag{11.2-2}$$

The position vector can also be expressed in vector notation. This is accomplished by assigning a unit vector along the positive coordinate direction. Assuming that the vector $\underline{i}$ is associated with the x-axis, the position vectors in the above example may be written in vector notation as

$$\underline{x}_1 = 4\ \underline{i}\ m \qquad and \qquad \underline{x}_2 = -3\ \underline{i}\ m \quad . \tag{11.2-3}$$

Referring to Figure 2, the position vector, $\underline{x}$, to an arbitrary point P located at position x is

$$\underline{x} = x\ \underline{i} \quad . \tag{11.2-4}$$

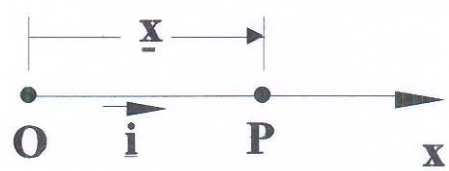


Figure 2. Position of a particle.

Definition: Displacement Vector

In general, the *displacement vector* is defined as the change in the position of a point. Referring to Figure 3, consider a point as it travels from point P to P'.

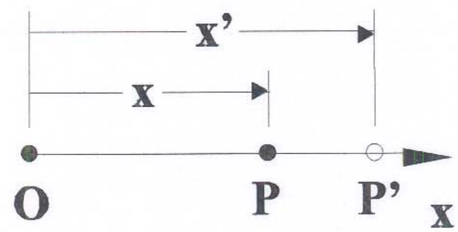


Figure 3. Displacement of a particle.

The displacement vector can be expressed in scalar notation as

$$\Delta x = x' - x \tag{11.2-5}$$

where Δx is positive when $x' > x$ and negative when $x' < x$.

In vector notation, the displacement vector, $\underline{d}$, can be written as

$$\underline{d} = \Delta x \, \underline{i} \quad . \tag{11.2-6}$$

Definition: Average Velocity; Instantaneous Velocity; Speed

Referring to Figure 4, consider a particle that moves through displacement Δx from point $P(x,t)$ to point $P'(x+\Delta x, t+\Delta t)$ during the time interval Δt.

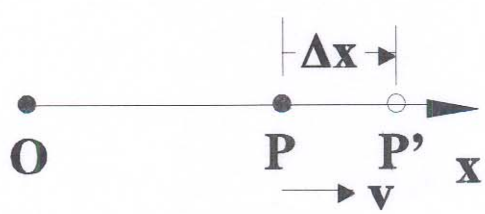


Figure 4. Velocity of a particle.

The *average velocity* of the particle is

$$v_{avg} = \frac{\Delta x}{\Delta t} \quad . \tag{11.2-7}$$

The *instantaneous velocity* is defined by letting Δt go to zero and may be expressed in scalar notation as

$$\boxed{v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x} \quad .} \tag{11.2-8}$$

In vector notation, the quantity is

$$\underline{v} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\, \underline{i} = \frac{dx}{dt}\, \underline{i} \quad . \tag{11.2-9}$$

In general, the direction of the velocity vector is tangent to the path and, for rectilinear motion, is constant. The magnitude of the velocity vector is called the *speed* of the particle.

From Equation (11.2-8), it is apparent that the velocity at any instant in time is the slope of the x vs. t curve. Since dt is always positive, the sign of v is that of dx. Thus, a negative velocity indicates that the particle is moving in the negative x direction.

The units of velocity are ft/sec (U.S.) and m/sec (MKS). The conversion between these units is 1 ft/sec = 0.3048 m/sec.

Example: A particle moves at a constant speed of 3 ft/sec along the negative x direction. Express the velocity vector in scalar and vector notation.

Since the movement is along negative x, the scalar expression is simply v = - 3 ft/sec. The vector notation includes the unit vector and is given by $\underline{v}$ = - 3 $\underline{i}$ ft/sec.

Definition: Average Acceleration, Instantaneous Acceleration

Referring to Figure 5, consider a particle moving with velocity v at time equal to t; P(v,t). After a Δt, the particle moves to P' at which time it has a velocity v' = v+Δv; P'(v+Δv, t+Δt).

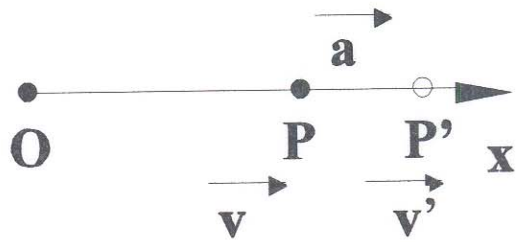


Figure 5. Acceleration of a particle.

The *average acceleration* of the particle is

$$a_{avg} = \frac{\Delta v}{\Delta t} \quad . \qquad \textbf{(11.2-10)}$$

The *instantaneous acceleration* is defined by letting Δt go to zero and may be expressed in scalar notation as

$$\boxed{a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x} \quad .} \qquad \textbf{(11.2-11)}$$

From Equation (11.2-8), dt = dx/v. Substituting the latter into Equation (11.2-11),

$$\boxed{a = v \frac{dv}{dx} \quad .} \qquad \textbf{(11.2-12)}$$

In vector notation, the quantity is

$$\underline{a} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \underline{i} = \frac{dv}{dt} \underline{i} = \frac{dx^2}{dt^2} \underline{i} = v \frac{dv}{dx} \underline{i} \quad . \qquad \textbf{(11.2-13)}$$

In rectilinear motion, the direction of the acceleration vector is tangent to the path. However, in the more general case of curvilinear motion, this is not the case.

From Equations (11.2-11) and (11.2-12), it is apparent that the acceleration at any instant in time is the slope of the v vs. t curve. Since dt is always positive, the sign of a is that of dv. Thus, for

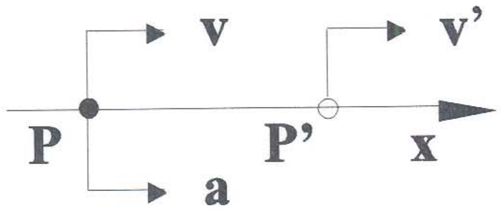


Figure 6. A particle accelerates with a > 0.

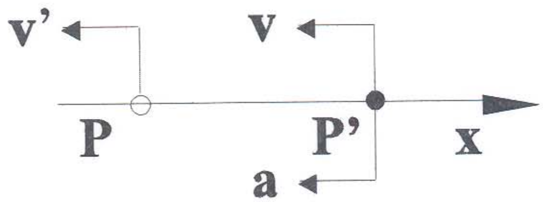


Figure 7. A particle decelerates with a > 0.

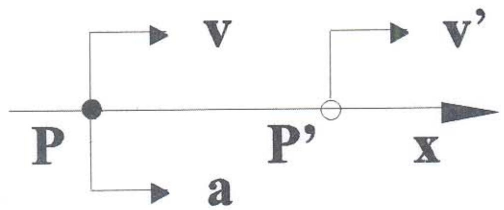


Figure 8. A particle decelerates with a < 0.

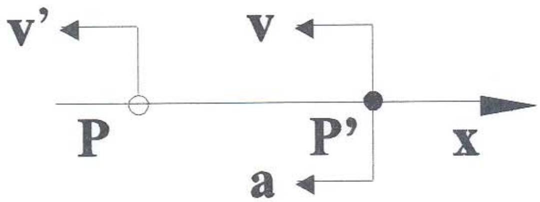


Figure 9. A particle accelerates with a < 0.

rectilinear motion, a negative acceleration indicates that the speed of the particle is decreasing while a positive acceleration indicates that the speed of the particle is increasing.

The units of acceleration are ft/sec^2 (U.S.) and m/sec^2 (MKS). The conversion between these units is 1 ft/sec^2 = 0.3048 m/sec^2.

Nomenclature:

The *nomenclature* used in conjunction with acceleration may be confusing.

Consider, for example, the case depicted in Figure 6 in which v = 4 m/sec; v' = 6 m/sec; t = 2 sec. In this case, $\Delta v = v' - v = 2$ m/sec. Equation (11.2-11) shows that a = 1 m/sec^2 when $\Delta t = 2$ sec. In this case, a is positive and, since the final speed of the particle (absolute value of v') is greater than the initial speed (absolute value of v), the particle is said to *accelerate*. This all seems to make sense.

However, as illustrated in Figure 7, when v = - 6 m/sec and v' = - 4 m/sec, a is still positive and equal to 1 m/sec^2 from Equation (11.2-11). However, since the particle is slowing down, the particle is said to *decelerate*.

A similar situation occurs when v = 6 m/sec and v' = 4 m/sec. In this case, depicted in Figure 8, a = - 1 m/sec^2. Since the particle is slowing down it is said to *decelerate*. Again, this seems to make sense.

However, as illustrated in Figure 9, when v = - 4 m/sec and v' = - 6 m/sec, a is still equal to - 1 m/sec^2 but, since the particle speeds up, it is said to *accelerate*.

Hence, it is possible to speak of a particle being accelerated or decelerated (depending upon whether it is slowing down or speeding up) but the acceleration itself ($\underline{a} = d\underline{v}/dt$) can be either positive or negative.

Example: See Sample Problem 11.1 on page 589 of the 6^{th} edition of Beer and Johnson.

11.3 Motion of A Particle

The motion of a particle is seldom defined by a relation between x and t (where one can simply differentiate with respect to x to find v and a) but rather by its acceleration. In this case, it is necessary to integrate to determine v and x. Several different cases must be studied.

Equation (11.3-1) illustrates the first case in which the acceleration is expressed as a function of time.

$$a = f(t) = \frac{dv}{dt} \quad . \tag{11.3-1}$$

To determine the velocity, the variables are separated

$$dv = a\ dt = f(t)\ dt \tag{11.3-2}$$

and by integrating

$$\int dv = \int a\ dt = \int f(t)\ dt \quad . \tag{11.3-3}$$

Since there are no limits on the integral, it is referred to as *indefinite* and a constant arises as a result of the integration. *Initial conditions* are introduced to avoid this problem. The integral then becomes *definite*. Consider, for example, the case in which

$$v(t_0) = v_0 \qquad and \qquad v(t) = v \quad . \tag{11.3-4}$$

In most cases, the initial time, t_0, is set equal to zero. However, in general,

$$\int_{v_0}^{v} dv = \int_{t_0}^{t} f(t)\ dt \tag{11.3-5}$$

and

$$v - v_0 = \int_{t_0}^{t} f(t)\ dt \qquad and \qquad v = v_0 + \int_{t_0}^{t} f(t)\ dt \quad . \tag{11.3-6}$$

A similar argument can be used to find the position, x. This may be accomplished by introducing Equation (11.2-8) and specifying initial conditions

$$x(t_0) = x_0 \qquad \textit{and} \qquad x(t) = x \quad . \tag{11.3-7}$$

After separating the variables and integrating

$$x - x_0 = \int_{t_0}^{t} v\,(t)\;dt \qquad \textit{and} \qquad x = x_0 + \int_{t_0}^{t} v\,(t)\;dt \quad . \tag{11.3-8}$$

Equation (11.3-9) illustrates the case in which the acceleration is expressed as a function of position; Equation (11.2-12) has been incorporate here.

$$a = f\,(x) = v\,\frac{dv}{dx} \quad . \tag{11.3-9}$$

Separating variables

$$v\;dv = a\;dx = f\,(x)\;dx \tag{11.3-10}$$

and integrating

$$\int_{v_0}^{v} v\;dv = \int_{x_0}^{x} f\,(x)\;dx \quad . \tag{11.3-11}$$

Solving,

$$\frac{1}{2}\,v^2 - \frac{1}{2}\,v_0^2 = \int_{x_0}^{x} f\,(x)\;dx \qquad \textit{and} \qquad v^2 = v_0^2 + 2\int_{x_0}^{x} f\,(x)\;dx \quad . \tag{11.3-12}$$

A similar argument can be used to find the position, x. This may be accomplished by introducing Equation (11.2-8) and solving for dt in terms of v as follows

$$dt = \frac{dx}{v} \tag{11.3-13}$$

where, in this case, v = v(x); the latter is determined from Equation (11.3-12).

By separating variables, integrating, and introducing initial conditions,

$$t = t_0 + \int_{x_0}^{x} \frac{dx}{v\,(x)} \quad . \tag{11.3-14}$$

Equation (11.3-15) illustrates the case in which the acceleration is expressed as a function of the velocity.

$$a = f(v) = \frac{dv}{dt} = v\,\frac{dv}{dx} \quad . \tag{11.3-15}$$

Separating variables

$$dt = \frac{dv}{f(v)} \qquad or \qquad dx = \frac{v\,dv}{f(v)} \quad . \tag{11.3-16}$$

Introducing the appropriate initial conditions

$$v(t_0) = v_0 \quad and \quad v(t) = v \qquad or \qquad v(x_0) = v_0 \quad and \quad v(x) = v \tag{11.3-17}$$

and integrating

$$t - t_0 = \int_{v_0}^{v} \frac{dv}{f(v)} \qquad or \qquad x - x_0 = \int_{v_0}^{v} \frac{v\,dv}{f(v)} \quad . \tag{11.3-18}$$

Then, one can use Equation (11.2-8) to determine the position.

A graphical solution may also be possible. In the case where position is plotted as a function of time, the velocity at any position is simply the slope of the position versus time curve [Equation (11.2-8)]. The acceleration at any point represents the slope of the velocity versus time curve [Equation (11.2-11)]. Alternatively, when acceleration is plotted as a function of time, the area under the curve represents the difference in velocity between the extremes [Equation (11.3-6)]. The area under the velocity versus time curve represents the difference in position at the extremes [Equation (11.3-8)].

11.4 Uniform and Uniformly Accelerated Rectilinear Motion

Uniform rectilinear motion occurs when the acceleration is equal to zero for all values of time. In this case,

$$a = \frac{dv}{dt} = 0 \quad . \tag{11.4-1}$$

Assuming $v(t_0) = v_0$ [at time $t = t_0$, $v = v_0$],

$$\int_{v_0}^{v} dv = 0 \qquad (11.4\text{-}2)$$

and

$$\boxed{v - v_0 = 0 \qquad and \qquad v = v_0 = constant \ .} \qquad (11.4\text{-}3)$$

Using Equation (11.2-8)

$$v = \frac{dx}{dt} = v_o \ . \qquad (11.4\text{-}4)$$

Separating variables and integrating with the initial conditions, $x(t_0) = x_0$ and $x(t) = x$,

$$\boxed{x - x_0 = \int_{t_0}^{t} v\,(t)\,dt = v_0 \int_{t_0}^{t} dt \qquad and \qquad x = x_0 + v_0\,(t - t_0) \ .} \qquad (11.4\text{-}5)$$

Note in Equation (11.4-5) that v_0 can be taken outside of the integral sign because it is a constant.

Uniformly accelerated rectilinear motion, on the other hand, occurs when the acceleration is a constant. In this case,

$$a = \frac{dv}{dt} = a_0 \ . \qquad (11.4\text{-}6)$$

Thus,

$$\int_{v_0}^{v} dv = \int_{t_0}^{t} a_0\,dt = a_0 \int_{t_0}^{t} dt \qquad (11.4\text{-}7)$$

and

$$\boxed{v - v_0 = a_0\,(t - t_0) \qquad and \qquad v = v_0 + a_0\,(t - t_0) \ .} \qquad (11.4\text{-}8)$$

Recalling from Equation (11.2-8) that $dx/dt = v$, and using v as given by Equation (11.4-8),

$$x - x_0 = \int_{t_0}^{t} [\, v_0 + a_0 (t - t_0) \,]\, dt \ \text{and} \ x = x_0 + v_0 (t - t_0) + \frac{1}{2} a_0 (t - t_0)^2 \quad . \tag{11.4-9}$$

Also recall that

$$v \frac{dv}{dx} = a = a_0 \quad . \tag{11.4-10}$$

Integrating

$$\int_{v_0}^{v} v\, dv = \int_{x_0}^{x} a_0\, dx = a_0 \int_{x_0}^{x} dx \tag{11.4-11}$$

and

$$\frac{1}{2} v^2 - \frac{1}{2} v_0^2 = a_0 (x - x_0) \qquad \text{and} \qquad v^2 = v_0^2 + 2\, a_0 (x - x_0) \quad . \tag{11.4-12}$$

A freely falling body is an excellent example of uniformly accelerated rectilinear motion where

$$a = g = 9.81 \frac{m}{\sec^2} = 32.2 \frac{ft}{\sec^2} \quad . \tag{11.4-13}$$

Example: See Sample Problem 11.4 on page 600 of the 6^{th} edition of Beer and Johnson. Note that $a_y = - 9.81\ m/s^2$, since positive y is assumed upward; the relative velocity concept is covered in the next section.

11.5 Relative and Dependent Motion

Figure 10 shows two particles moving along the same straight line.

The *relative position* coordinate of B with respect to A is given by $x_{B/A}$ where

$$x_{B/A} = x_B - x_A \tag{11.5-1}$$

and

$$\boxed{x_B = x_A + x_{B/A}\ .} \tag{11.5-2}$$

For $x_{B/A} > 0$, an observer located at A would see B along positive x; if $x_{B/A} < 0$, the observer would see B along negative x.

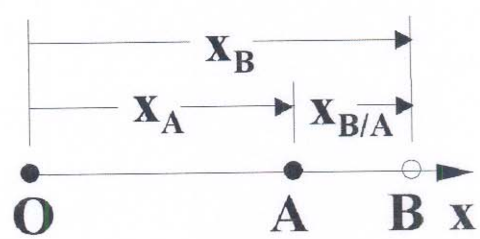


Figure 10. Relative position of B wrt A.

The *relative velocity* of B with respect to A, $v_{B/A}$, can be found by differentiating Equation (11.5-1) as follows:

$$v_{B/A} = v_B - v_A \tag{11.5-3}$$

and

$$\boxed{v_B = v_A + v_{B/A}\ .} \tag{11.5-4}$$

The *relative acceleration* of B with respect to A is determined by differentiating Equation (11.5-3) as

$$a_{B/A} = a_B - a_A \tag{11.5-5}$$

and

$$a_B = a_A + a_{B/A} \quad . \tag{11.5-6}$$

In these formulas, point A is considered the reference point from which B is observed. The concept becomes very important in future considerations for studying the motion of systems of particles and rigid bodies. It is studied in more detail for the more general case of curvilinear motion later in this chapter.

Dependent motion takes place when the motion of one particle depends upon the motion of another. Consider, for example, the configuration depicted in Figure 11 where the positions of the pulleys and mass are specified by the coordinates x_A, x_B, and x_C.

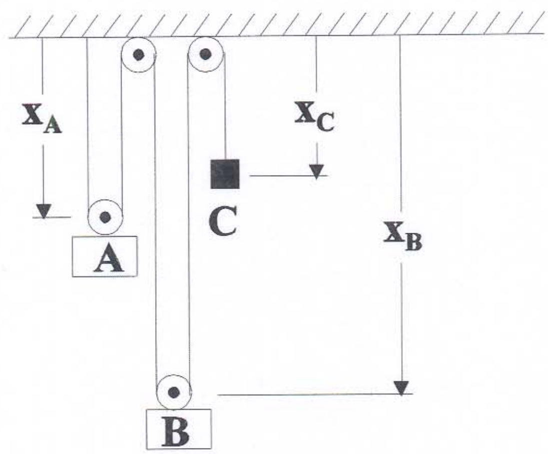


Figure 11. Dependent motion.

In this case, the length of the string, L, remains constant necessitating that

$$2\, x_A + 2\, x_B + x_C = L = constant \quad . \tag{11.5-7}$$

Hence, although there are three coordinates labeled on the figure, there is one dependent relation. Therefore, any two of the coordinates may be arbitrarily chosen to completely define the motion. These are referred to as the "*degrees of freedom.*"

When a dependent relation, such as that shown in Equation (11.5-7), is established, similar relations can be established for the velocities and accelerations by direct differentiation. In the case described by Equation (11.5-7)

$$2\ v_A + 2\ v_B + v_C = 0 \tag{11.5-8}$$

and

$$2\ a_A + 2\ a_B + a_C = 0\ . \tag{11.5-9}$$

Example: As shown in Figure 12, an elevator, E, moves downward at a constant velocity of 15 ft/sec. W is a counterweight and C is a point fixed to the cable. Determine v_C, v_W, $v_{C/E}$, and $v_{W/E}$.

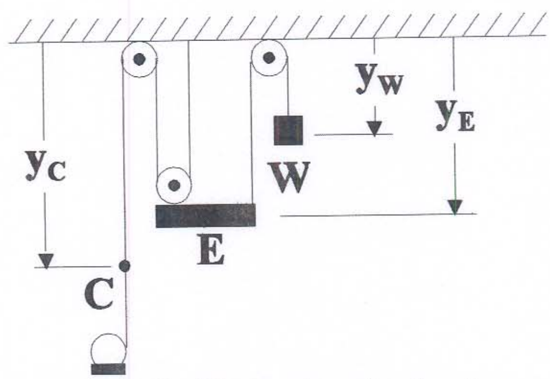


Figure 12. Kinematics of a elevator.

Since the distance between point C and the end of the cable remains constant

$$y_C + 2\ y_E = C_1 = constant\ . \tag{1}$$

Differentiating gives the required velocity relation

$$v_C + 2\ v_E = 0\ . \tag{2}$$

Substituting $v_E = 15$ ft/sec into the above equation yields

$$\boxed{v_C = -\ 30\ ft/\sec = -\ 9.14\ m/\sec\ .} \tag{3}$$

Since the positive coordinate is measured downward, the negative sign in the velocity implies that C moves upward. The factor used to convert the U.S. units to MKS is

$$1\ ft = 0.3048\ m\ . \tag{4}$$

Since the cable associated with the counterweight also remains constant

$$y_E + y_W = C_2 = constant \quad . \tag{5}$$

Differentiating Equation (5)

$$v_E + v_W = 0 \quad . \tag{6}$$

Substituting v_E = 15 ft/sec into the above equation yields

$$\boxed{v_W = -\ 15\ ft/\text{sec} = -\ 4.57\ m/\text{sec} \quad .} \tag{7}$$

The relative velocity of C with respect to E is expressed in accordance with the expression given in Equation (11.5-3) as

$$\boxed{v_{C/E} = v_C - v_E = -\ 30 - (15) = -\ 45\ ft/\text{sec} = -\ 13.72\ m/\text{sec} \quad .} \tag{8}$$

A similar argument can be used to determine the relative velocity of W with respect to E as

$$\boxed{v_{W/E} = v_W - v_E = -\ 15 - (15) = -\ 30\ ft/\text{sec} = -\ 9.14\ m/\text{sec} \quad .} \tag{9}$$

Note how the final answers have been boxed and converted.

11.6 Curvilinear Motion - Rectangular Coordinates

When a particle moves along an arbitrary path in space it is said to be in *curvilinear motion*. When the motion is confined to a single plane, it is referred to as *plane motion*. If the particle moves in space, the motion is referred to as *three-dimensional motion*. A number of different coordinate systems can be used to describe these situations but the Cartesian system is used most frequently.

Figure 13 depicts the motion of a particle referred to a Cartesian reference system.

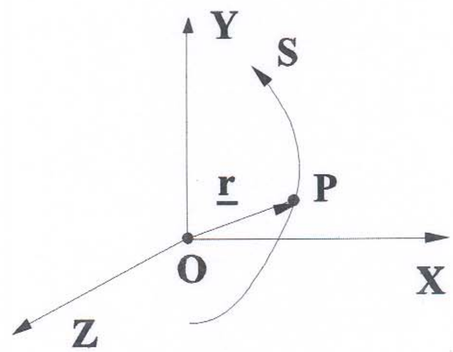


Figure 13. A particle in space.

In this case, the position vector, $\underline{r}$, may be expressed in terms of vector notation as

$$\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k} \qquad \textbf{(11.6-1)}$$

where i, j, and k are unit vectors directed along x, y, and z, respectively.

The expression for the velocity is found by differentiating Equation (11.6-1) as

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{x}\,\underline{i} + \dot{y}\,\underline{j} + \dot{z}\,\underline{k} = v_x\,\underline{i} + v_y\,\underline{j} + v_z\,\underline{k} \ . \qquad \textbf{(11.6-2)}$$

A positive value for v_x indicates that the vector component, $\underline{v}_x$, is directed to the right and a negative value indicates that it is directed to the left. The sense of the other components can be determined in a similar fashion.

The acceleration is found by differentiating Equation (11.6-2) as

$$\underline{a} = \frac{d\underline{v}}{dt} = \ddot{x}\,\underline{i} + \ddot{y}\,\underline{j} + \ddot{z}\,\underline{k} = \dot{v}_x\,\underline{i} + \dot{v}_y\,\underline{j} + \dot{v}_z\,\underline{k} = a_x\,\underline{i} + a_y\,\underline{j} + a_z\,\underline{k} \ . \qquad \textbf{(11.6-3)}$$

11.7 Projectile Motion

The use of rectangular components to describe the position, velocity, and acceleration of a particle is particularly effective when the component a_x of the acceleration depends only upon t, x, and or v_x, and when the component a_y depends only upon t, y, or v_y, and az upon t, z, and v_z. The plane motion of a projectile is an important example of this situation.

In projectile motion, a two-dimensional coordinate system is established, usually with x to the right and y upward. Assuming that the particle is subjected to a vertical gravitational field with no air resistance, the motion along x is uniform while the motion along y is uniformly accelerated (by the action of gravity). The governing equations for this case are

$$a_x = 0 \qquad and \qquad a_y = -g \tag{11.7-1}$$

where g = 9.81 m/s^2 = 32.2 ft/s^2.

As described in Section 11.4, the velocity and position are determined by separating variables and integrating, and are given by

$$v_x = v_{0_x} \qquad and \qquad v_y = v_{0_y} - g\,t \tag{11.7-2}$$

and

$$x = x_0 + v_{0_x}\,t \qquad and \qquad y = y_0 + v_{0_y}\,t - \frac{1}{2}\,g\,t^2 \ . \tag{11.7-3}$$

It is important to include the subscripts in these equations. The terms that include the subscript "0" correspond to the initial position and/or velocity of the particle.

Example: A little league baseball player hits a ball over a 12 ft high outfield wall located at a distance of 200 ft from home plate. A video tape showed that the batter's hands were 3 ft above the ground when she hit the ball at an angle of inclination of 30^0 with respect to the horizontal. Determine (a) the initial velocity of the ball when it left the bat and (b) the maximum height reached by the ball.

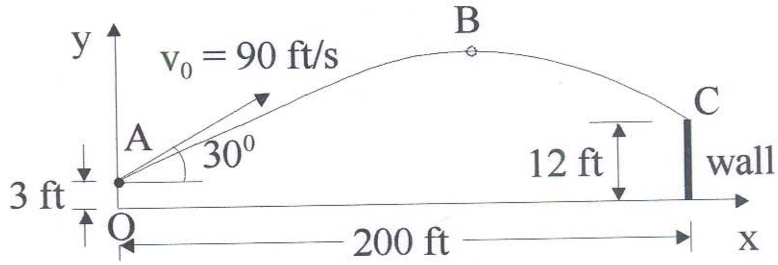


Figure 14. A ball is ht over an outfield wall.

(a) Referring to Figure 14,

$$x_0 = 0 \qquad y_0 = 3 \qquad v_{0_x} = v_0 \cos 30^o \qquad v_{0_y} = v_0 \sin 30^o \quad . \tag{1}$$

Using the values in Equation (1) and substituting into the first expression in Equation (11.7-3) for point C where x = 200,

$$200 = 0 + v_0 \cos 30^o \qquad or \qquad t = \frac{230.94}{v_0} \quad . \tag{2}$$

Substituting the latter into the second expression in Equation (11.7-3), noting that y = 12 at point C,

$$12 = 3 + v_0 \sin 30^o \frac{230.94}{v_0} - \frac{32.2}{2} [\frac{230.94}{v_0}]^2 \qquad or \qquad v_0^2 = 12{,}800 \tag{3}$$

and

$$\boxed{v_0 = 89.9 \frac{ft}{s} = 27.4 \frac{m}{s} \quad .} \tag{4}$$

(b) At point B, $v_y = 0$. Substituting the latter into the second expression in Equation (11.7-2) and using the initial velocity specified in Equation (4),

$$0 = 89.8 \sin 30^o - 32.2\, t \qquad or \qquad t = 1.394\ s \quad . \tag{5}$$

Substituting the value of t found in Equation (5) into the second expression in Equation (11.7-3),

$$y = 3 + 89.8 \sin 30^o (1.394) - \frac{32.2}{2} (1.394)^2 \tag{6}$$

and

$$\boxed{y = 34.4\ ft = 10.5\ m \quad .} \tag{7}$$

11.8 Velocity and Acceleration in Curvilinear Motion

Figure 15 shows a particle that moves through along a curvilinear path through a distance ΔS from point P (given by the position vector $\underline{r}$) to P' (given by position vector $\underline{r}'$). The *average velocity* of the particle can be expressed in terms of the change in the position vector, $\Delta\underline{r}$, as

$$\underline{v}_{avg} = \frac{\Delta \underline{r}}{\Delta t} \quad . \tag{11.8-1}$$

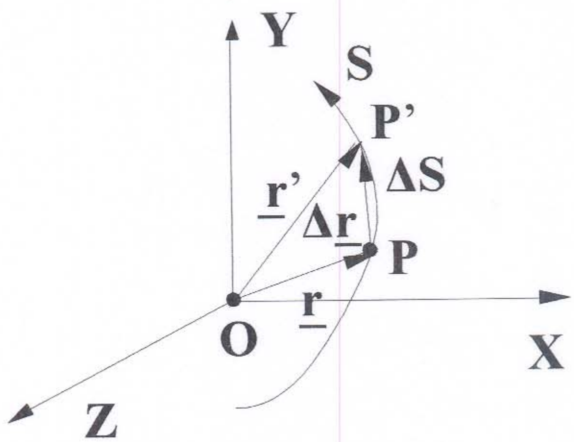


Figure 15. A particle in curvilinear motion.

Equation (11.8-1) is a vector equality and shows that the velocity vector is directed in the same direction as $\Delta\underline{r}$. The *instantaneous velocity* is given by

$$\boxed{\underline{v} = = \lim_{\Delta t \to 0} \frac{\Delta \underline{r}}{\Delta t} = \frac{d \underline{r}}{d t} \quad .} \tag{11.8-2}$$

Similarly, the *average acceleration* is given by

$$\underline{a}_{avg} = \frac{\Delta \underline{v}}{\Delta t} \quad . \tag{11.8-3}$$

Equation (11.8-3) shows that the acceleration vector is in the direction of $\Delta\underline{v}$ (not $\underline{v}$ or $\Delta\underline{r}$) and, in general, is not tangent to the path when the direction of the velocity vector changes. However, in the case of the rectilinear motion studied earlier, $\underline{a}$, was along the direction of motion. The *instantaneous acceleration* is given by

$$\underline{a} = = \lim_{\Delta t \to 0} \frac{\Delta \underline{v}}{\Delta t} = \frac{d \underline{v}}{d t} . \qquad \textbf{(11.8-4)}$$

11.9 Relative Motion in Curvilinear Coordinates

Figure 16 depicts the movement of two particles A and B referred to two axes systems that are in translation (the origin of the axes move with respect to one another but the individual axes remain parallel to one another).

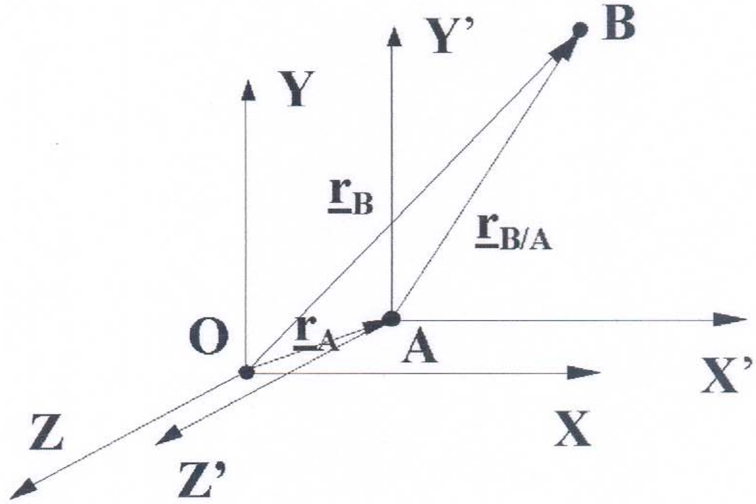


Figure 16. Relative motion of particles.

The position vectors to the particles measured relative to a global reference system located at point O are denoted by $\underline{r}_A$ and $\underline{r}_B$, respectively. The position vector of B as seen from an observer standing at the reference point A, $\underline{r}_{B/A}$, is given by the relation

$$\underline{r}_B = \underline{r}_A + \underline{r}_{B/A} \qquad \textbf{(11.9-1)}$$

while the corresponding scalar components are

$$x_B = x_A + x_{B/A} \qquad and \qquad y_B = y_A + y_{B/A} . \qquad \textbf{(11.9-2)}$$

Also,

$$\boxed{\underline{v}_B = \underline{v}_A + \underline{v}_{B/A}} \tag{11.9-3}$$

or

$$\dot{x}_B = \dot{x}_A + \dot{x}_{B/A} \qquad and \qquad \dot{y}_B = \dot{y}_A + \dot{y}_{B/A} \tag{11.9-4}$$

and

$$\boxed{\underline{a}_B = \underline{a}_A + \underline{a}_{B/A}} \tag{11.9-5}$$

or

$$x_B = \ddot{x}_A + \ddot{x}_{B/A} \qquad and \qquad \ddot{y}_B = \ddot{y}_A + \ddot{y}_{B/A} \ . \tag{11.9-6}$$

The motion of B with respect to the global system [x,y,z] is called the absolute motion of B whereas the motion of B with respect to [x',y',z'] is the relative motion of B with respect to A. Hence, the absolute motion of B can be expressed in terms of the absolute motion of A and the relative motion of B with respect to A. As mentioned previously, this concept is very important when analyzing the motion of rigid bodies.

11.10 Normal and Tangential Components

From the above discussion it is apparent that the velocity vector is tangent to the path but the acceleration vector is not. It is sometimes convenient to resolve the acceleration vector into components normal and tangent to the path. As illustrated in Figure 17, coordinates are path dependent and move along with the particle.

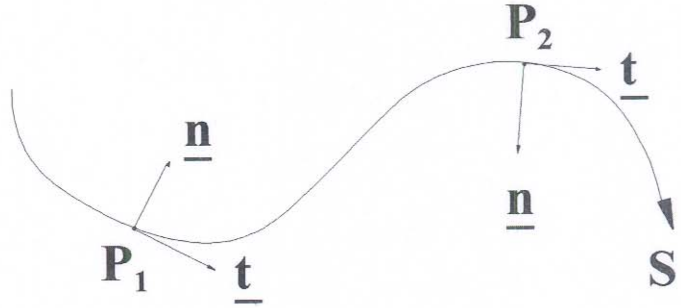


Figure 17. Tangential and normal components.

The positive tangent direction is chosen to coincide with the direction of motion while the positive normal direction is chosen directed toward the instantaneous center of curvature.

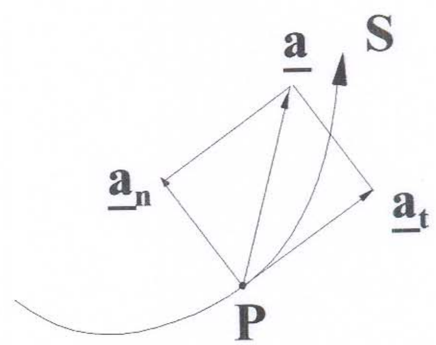


Figure 18. Acceleration components.

Figure 18 shows the acceleration components given by

$$\underline{a} = = \frac{v^2}{\rho}\,\underline{n} + \frac{dv}{dt}\,\underline{t} = \frac{v^2}{\rho}\,\underline{n} + v\,\frac{dv}{ds}\,\underline{t} \qquad \textbf{(11.10-1)}$$

where s is the arc length, dv/ds is the speed of the particle as it moves along the path, and ρ is the radius of curvature.

The normal component reflects a change in the direction of the motion associated with ρ while the tangential component reflects a change in speed. The acceleration is zero if and only if both components vanish.

It is important to realize that a particle moving at constant speed will not have an acceleration equal to zero unless the particle happens to pass through a point of inflection (where $\rho = \infty$) or is moving along a straight line.

Figure 19 shows the special case in which a particle moves along a circular path with constant $\rho = r$. In this special case, the arc length s is given by

$$s = r\,\theta \qquad \textbf{(11.10-2)}$$

and, since r is constant,

$$v = \frac{ds}{dt} = \frac{d}{dt}(r\,\theta) = r\,\dot{\theta} + \dot{r}\,\theta = r\,\dot{\theta} \quad . \tag{11.10-3}$$

By substituting Equation (11.10-3) into Equation (11.10-1), again realizing that r is constant,

$$\underline{a} = = \frac{v^2}{\rho}\,\underline{n} + \frac{dv}{dt}\,\underline{t} = r\,\dot{\theta}^2\,\underline{n} + \dot{v}\,\underline{t} = v\,\dot{\theta}\,\underline{n} + r\,\ddot{\theta}\,\underline{t} \tag{11.10-4}$$

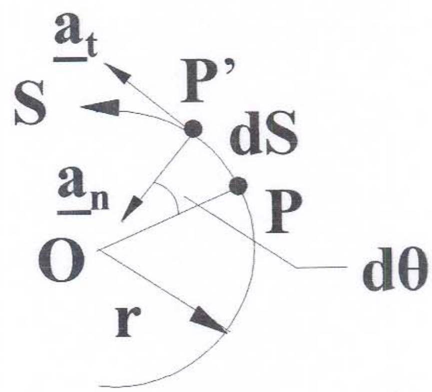


Figure 19. A particle moving in a circle.

Example: An outdoor track is 420 ft in diameter. A runner increases her speed at a constant rate from 15 to 25 ft/s over a distance of 100 ft. Determine the total acceleration of the runner 2.5 seconds after she begins to increase her speed.

Assume that at point A the runner is moving at a speed of 15 ft/s. Point B lies at a distance of 100 ft along the track from A where she reaches a speed of 25 ft/s. Assuming that the acceleration is constant over the distance between points A and B,

$$a_t = \frac{dv}{dt} = v\,\frac{dv}{ds} \quad . \tag{1}$$

Since the distance is specified, as opposed to the time required to change the velocity, the term on the right hand side of Equation (1) is utilized to determine a_t. Separating variables and integrating,

$$v^2 = v_0^2 + 2\, a_t\, (s - s_0) \qquad or \qquad (25)^2 = (15)^2 + 2\,(100)\, a_t \tag{2}$$

and

$$a_t = 2\, \frac{ft}{s^2} \; . \tag{3}$$

To determine the normal component of the acceleration, a_n, 2.5 seconds after the speed is increased, it is necessary to determine the velocity at that time. This is done by using the value of a_t specified in Equation (3) and the second form of a_t in Equation (1) as,

$$v = v_0 + a_t\, t = 15 + 2\,(2.5) = 20\, \frac{ft}{s} \; . \tag{4}$$

Thus, when t = 2 seconds, the runner is at a point between A and B where

$$a_n = \frac{v^2}{\rho} = \frac{(20)^2}{210} = 1.9\, \frac{ft}{s^2} \tag{5}$$

and

$$\boxed{a = \sqrt{a_t^2 + a_n^2} = \sqrt{(2)^2 + (1.9)^2} = 2.76\, \frac{ft}{s^2} = 0.84\, \frac{m}{s^2} \; .} \tag{6}$$

11.11 Radial and Transverse Components

It is sometimes convenient to use polar coordinates. In this case, the velocity and acceleration of the particle can be resolved into radial and transverse components directed along and perpendicular, respectively, to the position vector of the particle. Referring to Figure 20,

$$\boxed{v_r = \frac{dr}{dt} = \dot{r}} \tag{11.11-1}$$

and

$$v_\theta = r\,\frac{d\theta}{dt} = r\,\dot{\theta} \quad . \tag{11.11-2}$$

The acceleration vector can also be resolved into radial and transverse components as follows

$$a_r = \frac{dv_r}{dt} - v_\theta\,\frac{d\theta}{dt} = \dot{v}_r - v_\theta\,\dot{\theta} = \ddot{r} - r\,\dot{\theta}^2$$
$$a_\theta = \frac{dv_\theta}{dt} + v_r\,\frac{d\theta}{dt} = \dot{v}_\theta + v_r\,\dot{\theta} = r\,\ddot{\theta} + \dot{r}\,\dot{\theta} + \dot{r}\,\dot{\theta} = r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} \quad . \tag{11.11-3}$$

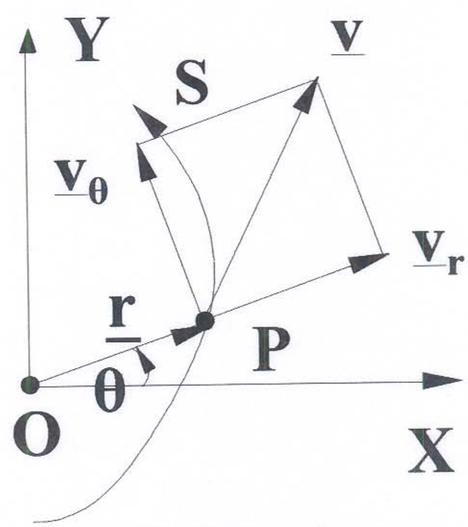


Figure 20. Radial and transverse components.

For the special case of motion along a circular path (r = constant), dr/dt and d^2r/dt^2 are zero, so

$$v_r = 0 \qquad and \qquad v_\theta = r\,\dot{\theta} \tag{11.11-4}$$

and

$$a_r = -\,r\,\dot{\theta}^2 \qquad and \qquad a_\theta = r\,\ddot{\theta} \quad . \tag{11.11-5}$$

For this special case, the components in Equation (11.11-6) are the same as those derived for the case of tangential and normal components except that the signs of the normal (directed inward toward the center of curvature) and radial (directed outward from the origin) components are different.

As shown later, polar coordinates may be used to describe the rotation of a rigid body about an axis. In this case, the motion is expressed in terms of the angular coordinate as

$$\theta = \theta(t) \quad . \tag{11.11-6}$$

The angular velocity, ω, and the angular acceleration, α, are defined as

$$\omega = \frac{d\theta}{dt} = \dot{\theta} \tag{11.11-7}$$

and

$$\alpha = \frac{d\omega}{dt} = \ddot{\theta} \tag{11.11-8}$$

respectively.

Example: See Sample Problem 11.12 on page 649 of the 6th edition of Beer and Johnson.

Chapter 12. Kinetics of Particles: Newton's Second Law

12.1 Introduction

Chapter 11 dealt with kinematics, the study of motion of a particle. This chapter discusses *kinetics*, the study of the relationship between the motion and what causes it.

There are three basic methods for solving problems in kinetics: use of work and energy principles, study of impulse and momentum, and direct application of Newton's second law. This chapter focuses on the latter.

12.2 Newton's Laws of Motion

Newton postulated three basic laws. His first law states that a particle originally at rest, or moving in a straight line with a constant velocity, will remain in that state provided that the particle is not subjected to an unbalanced force.

The second law dictates that when a particle that has a mass, m, is acted upon by an unbalanced force, $\underline{F}$, it experiences an acceleration, $\underline{a}$, that has the same direction as the force and a magnitude that is directly proportional to the force. This law is usually expressed by engineers as

$$\underline{F} = m\,\underline{a} \quad . \qquad \textbf{(12.2-1)}$$

Newton's third law states that for every force acting on a particle, the particle exerts an equal, opposite, and collinear reactive force.

Equation (12.2-1) shows that $\underline{F}$ and $\underline{a}$ are proportional and, since m is a scalar, are always in the same direction. When $\underline{F}$ is constant in both magnitude and direction, the particle travels in rectilinear motion and moves in the direction of the force at a constant velocity. When only the magnitude of the force changes, the particle continues to move along the direction of the force in a straight line but the particle accelerates (changes speed) as it moves. However, in the most general case when $\underline{F} = \underline{F}(t)$, the particle changes both direction and speed as it moves in curvilinear motion. In this case, $\underline{F}$ (and consequently $\underline{a}$) are not necessarily tangent to the path.

The quantities in Equation (12.2-1) must be referred to a newtonian frame of reference (inertial system) either fixed or moving with respect to the sun. The right hand side of the equation is often referred to as an *inertia term* and represents the particle's ability to resist changes in velocity. Thus, the *mass* of a particle is a quantitative measure of inertia.

12.3 Systems of Units

The units associated with the U.S. or MKS (SI) systems are specified by choosing three out of four of the following: force, mass, length, and time. The remaining quantity is derived on the basis of Equation (12.2-1).

In the MKS (SI) system, for example, the base units are length [measured in meters (m)], mass [kilograms (kg)], and time [seconds (s)]. The force, derived from Newton's second law, is measured in newtons (N). Applying Equation (12.2-1),

$$1\ N = (1\ kg)\ (1\ \frac{m}{s^2}) = 1\ \frac{kg\ m}{s^2} \quad . \tag{12.3-1}$$

Newton's second law can also be used to express the weight of a particle in terms of its mass. In this case, the weight is simply the force exerted on the particle by acceleration due to the action of gravity. By applying Equation (12.2-1) in the vertical direction

$$\boxed{w = m\ g} \tag{12.3-2}$$

where g is the local gravitational constant equal to 9.81 m/s^2 or 32.2 ft/s^2.

In the U.S. system, the base units are length [measured in feet (ft)], force [pounds (lb)], and time [seconds (s)]. The mass is derived from Equation (12.2-1) and expressed in slugs where

$$1\ slug = 1\ \frac{lb\ s^2}{ft} \quad . \tag{12.3-3}$$

The mass is obtained from the weight by rearranging Equation (12.3-2) as

$$m = \frac{w}{g} \quad . \tag{12.3-4}$$

It is important to note that the MKS system is characterized by base parameters that are independent of the location where measurements are taken. This is not true for the U.S. gravitational system.

The factors required to convert between the two systems are

$$1\ ft = 0.3048\ m \qquad 1\ lb = 4.448\ N \qquad 1\ slug = 14.59\ kg \quad . \tag{12.3-5}$$

12.4 Equations of Motion

The basic equation for a number of forces acting on a particle is,

$$\sum \underline{F} = m\ \underline{a} \quad . \tag{12.4-1}$$

The corresponding scalar equations of motion depend upon the coordinate system selected to define the motion. In the case of rectangular coordinates,

$$\sum F_x = m\ a_x \qquad \sum F_y = m\ a_y \qquad \sum F_z = m\ a_z \quad . \tag{12.4-2}$$

For normal and tangential coordinates,

$$\sum F_n = m\ a_n = m\ \frac{v^2}{\rho} \qquad \sum F_t = m\ a_t = m\ \frac{dv}{dt} = m\ v\ \frac{dv}{ds} \quad . \tag{12.4-3}$$

In equation (12.4-3), the normal force, F_n, is often referred to as the *centripetal* force.

For polar coordinates,

$$\sum F_r = m\ a_r = m\ (\ddot{r} - r\ \dot{\theta}^2) \qquad \sum F_\theta = m\ a_\theta = m\ (r\ \ddot{\theta} + 2\ \dot{r}\ \dot{\theta}) \quad . \tag{12.4-4}$$

The procedure that is usually followed to solve problems is:

1. Establish a convenient non-accelerating reference frame.
2. Draw the free-body diagram of the particle showing known and unknown forces.
3. Apply the appropriate equations of motion to determine the forces or accelerations.
4. If necessary, use the equations of kinematics to determine the velocity and/or position.

Example: See Sample Problems 12.1, 12.2, 12.5, and 12.6 on pages 676 and 679 of the 6th edition of Beer and Johnson.

12.5 Linear Momentum

Newton originally expressed his second law as

$$\sum \underline{F} = \frac{d}{dt}(m\,\underline{v}) = \dot{\underline{L}} \tag{12.5-1}$$

where the quantity m$\underline{v}$ (or $\underline{L}$), is called the *linear momentum*, expressed in terms of lb·s and kg·m/s in U.S. and MKS systems, respectively. Equation (12.4-1) is derived by applying the product rule for differentiation and making the assumption that mass is constant as follows:

$$\sum \underline{F} = \frac{d}{dt}(m\,\underline{v}) = m\,\frac{d\underline{v}}{dt} + \underline{v}\,\frac{dm}{dt} = m\,\underline{a} + 0 \ . \tag{12.5-2}$$

Equation (12.5-1) implies that when a particle is in equilibrium, the linear momentum of the particle remains constant, both in magnitude and direction. In the more general case, the study of momentum provides an alternate analysis to the application of Equation (12.4-1) and kinematics.

12.6 Angular Momentum

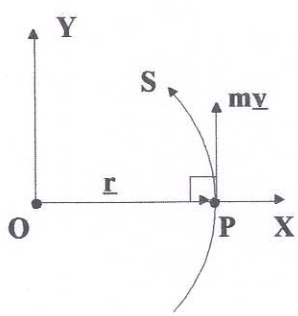


Figure 1. A particle moves along a circular path.

Figure 1 shows a particle moving along a circular path with constant speed, v. In this case, the linear momentum, $m\underline{v}$, is tangent to the path and perpendicular to the position vector, $\underline{r}$.

The moment of the linear momentum about a point is called the *angular momentum*, $\underline{H}$. In the case depicted in Figure 1, the angular momentum about the origin is

$$\underline{H}_O = \underline{r} \times m\, \underline{v} \qquad \textbf{(12.6-1)}$$

where the "x" symbol denotes the cross product operation. The units of angular momentum in U.S. and MKS systems are ft·lb·s and kg·m^2/s, respectively.

Figure 2 shows the more general case in which a particle moves along a curvilinear path in plane motion.

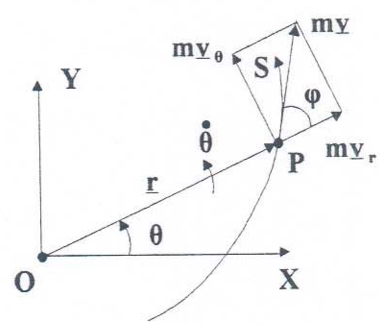


Figure 2. A particle in plane motion.

In this case, the linear momentum vector is broken up into polar components. Since the radial component passes through the origin, the angular momentum computed about that point depends only on the transverse component. In particular,

$$\underline{H}_O = \underline{r} \times m\, \underline{v} = r\, m\, v_\theta\, \underline{k} = = r\, m\, v \sin\varphi\, \underline{k} \qquad \textbf{(12.6-2)}$$

or using Equation (11.10-2),

$$\underline{H}_O = m\, r^2\, \dot{\theta}\, \underline{k} \ . \qquad \textbf{(12.6-3)}$$

12.7 Time Rate of Change of the Angular Momentum

Referring to Figure 2, the rate of change of the angular momentum about the origin is

$$\dot{\underline{H}}_O = \frac{d}{dt}\left(\underline{r} \ x \ m \ \underline{v} \right) = \dot{\underline{r}} \ x \ m \ \underline{v} + \underline{r} \ x \ m \ \dot{\underline{v}} \ . \qquad \textbf{(12.7-1)}$$

But the first term of the expression on the right hand side of the equation is zero since $d\underline{r}/dt$ (that is really equal to $\underline{v}$) and $m\underline{v}$ are collinear. Hence,

$$\dot{\underline{H}}_O = \underline{r} \ x \ m \ \underline{a} = \underline{r} \ x \ \underline{F} \qquad \textbf{(12.7-2)}$$

where $\underline{F}$ is the unbalanced force applied to the particle to create the motion. The right hand side of Equation (12.7-2) represents the moment of the force about the origin, $\underline{M}_O$.

If the particle were subjected to several concurrent forces,

$$\boxed{\sum \underline{M}_O = \dot{\underline{H}}_O \ .} \qquad \textbf{(12.7-3)}$$

Equation (12.7-3) shows that the rate of change of the angular momentum about the origin is equal to the total moment of the forces (applied to the particle) computed about the origin.

12. 8 Motion Under A Central Force

Figure 3 illustrates the case in which the only force acting on particle, P, is directed toward (or, alternatively, away) from a fixed point, O. In this case, the particle is said to be moving under the action of a *central force*; point O is referred to as the *center of force*.

Since the line of action of $\underline{F}$ passes through O, the moment of the force about this point is zero, and Equation (12.7-3) becomes

$$\dot{\underline{H}}_O = 0 \ . \qquad \textbf{(12.8-1)}$$

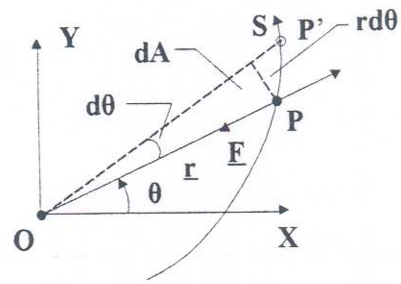


Figure 3. A particle moving under the action of a central force.

Therefore,

$$\underline{H}_O = constant \tag{12.8-2}$$

dictating that the angular momentum of a particle moving under the action of a central force is constant, in both magnitude and direction. From Equation (12.6-3), it is apparent that $mr^2\theta$ is also constant. Dividing the latter by the mass, provides a relation for the *angular momentum per unit mass*, h, as follows:

$$h = r^2\,\dot{\theta} = constant \quad . \tag{12.8-3}$$

The magnitude of h can be related to the *areal velocity* defined as the area swept out by the position vector to the particle per unit time (dA/dt). It can be shown using the nomenclature depicted on Figure 3 that the areal velocity is equal to $\frac{1}{2}\, r^2\theta$. The latter is related to h in the following equation:

$$Areal\ Velocity = \frac{1}{2}\, r^2\,\dot{\theta} = \frac{h}{2} = constant \quad . \tag{12.8-4}$$

Since the gravitational force exerted by the sun on a planet is a central force directed toward the center of the sun, Equations (12.8-2) and (12.8-4) are fundamental to the study of planetary motion. For a similar reason, they are also fundamental to study the motion of space vehicles in orbit about the earth.

12.9 Newton's Law of Gravitation

Figure 4 shows two masses, m and M, separated by a distance equal to r. When a mutual central force exists between the particles (created by gravitational attraction and/or electric or magnetic field effects) the forces are collinear and equal in magnitude. Furthermore,

$$F = \frac{G\,M\,m}{r^2} \tag{12.9-1}$$

where G is a universal constant called the constant of gravitation expressed in U.S. and MKS units as 3.442×10^{-8} ft^4/lb^4 and 6.673×10^{-11} m^3/kg^2.

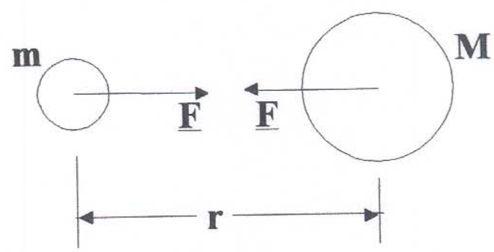


Figure 4. Mutual attraction of two particles.

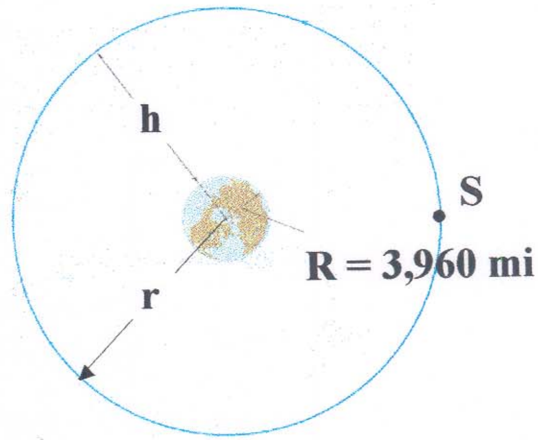
Gravitational forces exist between any pair of bodies, but their effect is appreciable only when one of the bodies has a large mass. As mentioned previously, the effect is pronounced for the cases of motion of a planet about the sun, of satellites orbiting about the earth, or bodies falling on the surface of the earth.

For example, the force exerted by the earth on a body of mass, m, located at or near the surface is defined as the weight, W, of the body. Substituting W = mg into Equation (12.9-1) reveals that

$$W = m\,g = \frac{G\,M\,m}{R^2} \quad or \quad G\,M = g\,R^2 \tag{12.9-2}$$

where M and R are the mass and radius of the earth. Since the earth is not round, g varies slightly; and, R is usually taken as the average value equal to 6.37×10^6 m (3,960 miles).

Example: Communication satellites are placed in a geosynchronous orbit so that they complete one full revolution about the axis of the earth in a sidereal day (23.934 h). The advantage of this is that the satellite appears stationary with respect to the ground.



(a) Prove that the satellite must be placed at an altitude of 22,240 mi above the surface of the earth.
(b) Determine the velocity at which these satellites must travel.

The period, $\tau = 23.934$ h $= 86.1624 \times 10^3$ s while the earth's radius R = 3960 mi = 20.9088×10^6 ft. Applying Newton's second law in the normal direction and using the the central force given in Equation (12.9-1),

$$F = \frac{G M m}{r^2} = m\, a_n = m \frac{v^2}{r} \qquad or \qquad v^2 = \frac{G M}{r} \quad . \tag{1}$$

After introducing the expression in Equation (12.9-2) into Equation (1)

$$v = R \sqrt{\frac{g}{r}} \quad . \tag{2}$$

Noting that the satellite travels at this speed through a distance of $2\pi r$ during the period, τ,

$$v\, \tau = 2\, \pi\, r \qquad or \qquad R \sqrt{\frac{g}{r}}\, \tau = 2\, \pi\, r \quad . \tag{3}$$

Solving Equation (3) for r,

$$r = \left[\frac{g\, \tau^2 R^2}{4\, \pi^2} \right]^{\frac{1}{3}} \quad . \tag{4}$$

But

$$h = r - R = \left[\frac{g\,\tau^2 R^2}{4\,\pi^2} \right]^{\frac{1}{3}} - R \tag{5}$$

$$= \left[\frac{32.2\ (86.1624x10^3)^2\ (20.9088x10^6)^2}{4\ \pi^2} \right] - 20.9088x10^6 = 118x10^6\ ft\ .$$

Or,

$$\boxed{h = 118x10^6\ ft = 22{,}240\ mi = 35.8x10^6\ m\ .} \tag{6}$$

From Equation (2),

$$\boxed{v = 20.9088x10^6 \sqrt{\frac{32.3}{38.3348x10^6}} = 10{,}087\ \frac{ft}{s} = 3{,}075\ \frac{m}{s}\ .} \tag{7}$$

Example: Assume that the satellite described in the previous problem is released from a space shuttle that is in a circular orbit at an altitude of 185 mi, and that the satellite is propelled by an upper-stage booster to its altitude of 22,240 mi.

As the satellite passes through point A (see Figure 5), the booster's motor is fired to inset the satellite into an elliptic transfer orbit. The booster is fired again at B to insert the satellite into the geosynchronous orbit. Knowing that the second firing increases the speed of the satellite by 4,810 ft/s, determine

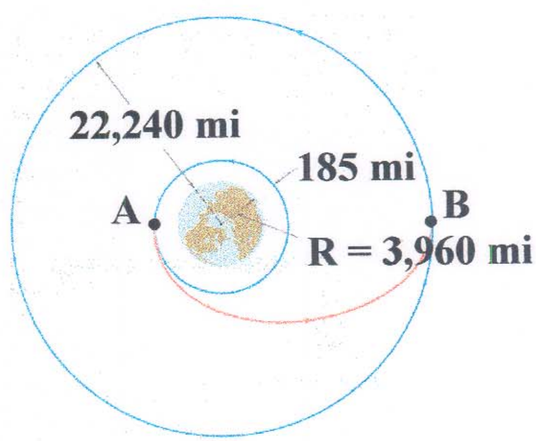


Figure 6. A satellite is launched from the Space Shuttle into geosynchronous orbit.

(a) the speed of the satellite as it approaches B on the elliptic transfer orbit.
(b) the increase in speed resulting from the first firing at A.

The distances from the center of the earth to points A and B are

$$r_A = (3{,}960 + 185)\ mi = 4{,}145\ mi = 21.886x10^6\ ft$$
$$r_B = (3{,}960 + 22{,}240)\ mi = 26{,}200\ mi = 138.336x10^6\ ft \quad . \tag{8}$$

Equation (2) can be applied to determine the velocities at points A and B when the satellite is traveling in a circular orbit as

$$v_{A\ circular}^2 = \frac{32.2\ (20.9088x10^6)^2}{21.8856x10^6} \quad and \quad v_{A_{circular}} = 25{,}362\ \frac{ft}{s}$$
$$v_{B\ circular}^2 = \frac{32.2\ (20.9088x10^6)^2}{138.336x10^6} \quad and \quad v_{A_{circular}} = 10{,}088\ \frac{ft}{s} \quad . \tag{9}$$

The relationship between the velocities in the circular and transfer orbits are

$$v_{A_{circular}} + \Delta v_A = v_{A_{transfer}}$$
$$v_{B_{transfer}} + \Delta v_B = v_{B_{circular}} \quad . \tag{10}$$

Applying the second relation in Equation (10) with the information in Equation (9) and $\Delta v_B = 4{,}810$ ft/s,

$$\boxed{v_{B_{transfer}} = v_{B_{circular}} - \Delta v_B = 10{,}088 - 4{,}810 = 5{,}278\ \frac{ft}{s} = 1{,}609\ \frac{m}{s} \quad .} \tag{11}$$

Finally, the conservation of angular momentum requires

$$r_A\ m\ v_{A_{transfer}} = r_B\ m\ v_{B_{transfer}} \quad or \quad v_{A_{transfer}} = \frac{26{,}200\ 5{,}278}{4{,}145} = 33{,}362\ \frac{ft}{s} \quad . \tag{12}$$

Using the first relation in Equation (10) and the information in Equations (9) and (12),

$$\boxed{\Delta v_A = v_{A_{transfer}} - v_{A_{circular}} = 33{,}362 - 23{,}362 = 8{,}000\ \frac{ft}{s} = 2{,}438\ \frac{m}{s} \quad .} \tag{13}$$

Chapter 13. Work and Energy

13.1 Work of a Force

The method of work and energy provides an alternative analysis to the application of Newton's second law and kinematics. Figure 1, for example, shows a particle, P, subjected to a force, $\underline{F}$, that has moved through an infinitesimal displacement, $d\underline{r}$.

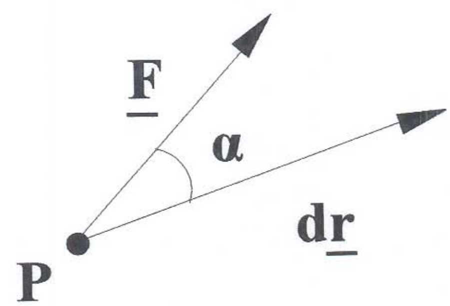


Figure 1. A particle that moves does work.

In a more global sense, the particle is moving along a curve so that the magnitude of the displacement vector is ds. As shown in the figure, the force makes an angle α with respect to the displacement vector. The increment of *work done by the force* is

$$dU = \underline{F} \cdot d\underline{r} = F\, ds \cos \alpha \quad . \qquad \textbf{(13.1-1)}$$

Equation (13.1-1) shows that work is the projection of $\underline{F}$ on $d\underline{r}$, or, conversely, the projection of $d\underline{r}$ on $\underline{F}$. If the projection and the vector have the same sense, the work is positive. When they are of opposite sign, the work is negative. The units associated with work are ft·lb (U.S.) and N·m (MKS).

As the particle moves between two points on the path, say s_1 and s_2, the total work may be expressed by performing an infinite sum of the infinitesimal work increments. This is accomplished mathematically by formulating a path integral; namely,

$$U_{1\rightarrow 2} = \int dU = \int_{s_1}^{s_2} F \cos \alpha \, ds \quad . \qquad \textbf{(13.1-2)}$$

When the force projection, $F \cos \alpha$, is plotted against the path coordinate, S, the work is simply the area under the curve between points s_1 and s_2.

Specific cases may help to quantify the argument. Figure 2, for example shows *one* of the unbalanced forces that act upon a particle that is moving in rectilinear motion (*i.e.*, R must be along x).

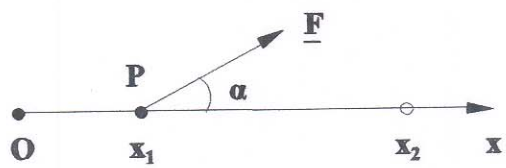


Figure 2. A point in rectilinear motion.

The work done while moving from position 1 to position 2 is

$$U_{1\to 2} = (F \cos \alpha)\,(x_2 - x_1) = (F \cos \alpha)\,\Delta x \quad . \qquad \textbf{(13.1-3)}$$

Figure 3, on the other hand, depicts an object moving upward from position y_1 to y_2 on a curved path under the action of gravity.

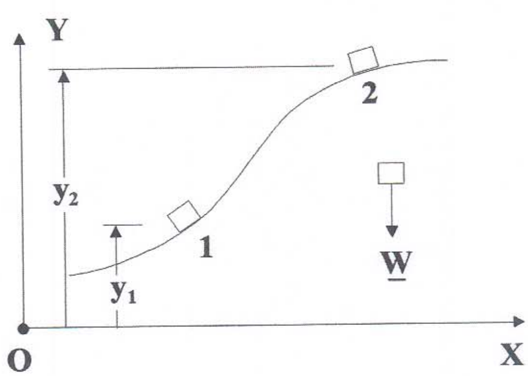


Figure 3. The weight of a particle does work as the particle moves upward.

In this case, the work done by the particle is its weight multiplied by the vertical displacement of its center of gravity. As shown in the insert, the force vector is directed opposite to the displacement, consequently, $dU = -W\,dy$. Since W is constant,

$$U_{1-2} = - W \int_{y_1}^{y_2} dy = - W (y_2 - y_1) = - W \Delta y \quad . \qquad \textbf{(13.1-4)}$$

Equation (13.1-4) shows that the work is positive when $\Delta y < 0$; that is, when the motion is directed downward.

Figure 4 shows a spring with a spring constant, k, expressed in terms of lb/ft (U.S.) or N/m (MKS).

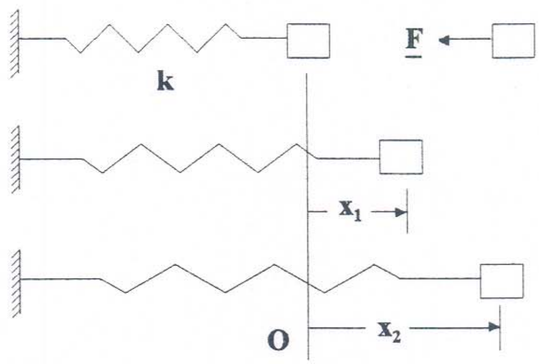


Figure 4. Work is done by a spring force.

As illustrated in the insert, the force generated by the spring is kx where x is measured from the undeformed position. The weight of the particle to which the spring is attached has been neglected in the formulation. Since the force is opposite to the displacement as the mass moves away from the deformed position, dU = - F dx, or, - (kx) dx, and

$$U_{1-2} = - k \int_{x_1}^{x_2} x \, dx = - \frac{k}{2} (x_2^2 - x_1^2) \quad . \qquad \textbf{(13.1-5)}$$

Equation (13.1-5) shows that the work is positive when $x_1 > x_2$; that is, when the spring is returning to its undeformed position.

Figure 5 shows the case where two particles of mass m and M are mutually attracted. It is assumed that the particle with mass M occupies a fixed position at O while the other particle moves along a curved path.

Focusing attention on the particle with mass m; since the force, $\underline{F}$, is directed toward point O, the work is negative as the position vector increases with dU = -F dr = -G (Mm/r^2) dr. Hence,

$$U_{1\text{-}2} = - G M m \int_{r_1}^{r_2} \frac{1}{r^2} dr = - G M m \left(\frac{1}{r_1} - \frac{1}{r_2}\right) . \qquad \textbf{(13.1-6)}$$

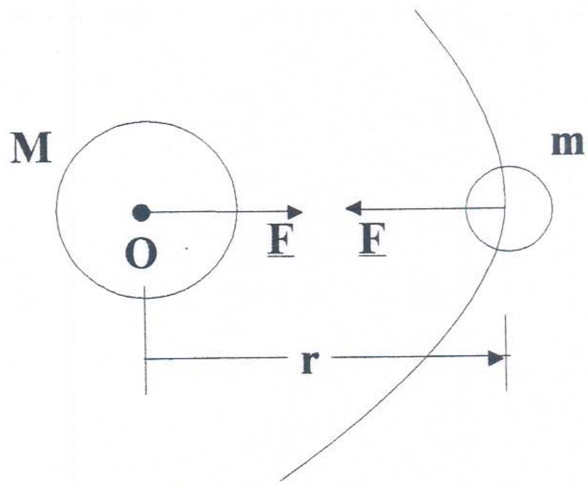


Figure 5. Work is done by a central force.

13.2 Kinetic Energy - Principle of Work and Energy

Figure 6 shows a force, F, separated into tangential and normal components. Since the normal component is perpendicular to the path, dr, the work that it does is zero. Thus, the work done by the tangential component represents the total work done. From Newton's second law

$$F_t = m \, a_t = m \, v \, \frac{dv}{ds} \qquad \textbf{(13.2-1)}$$

and

$$U = \int_{s_1}^{s_2} F \cos \alpha \, ds = m \int_{v_1}^{v_2} v \, dv = \frac{1}{2} m \, v_2^2 - \frac{1}{2} m \, v_1^2 \; . \qquad \textbf{(13.2-2)}$$

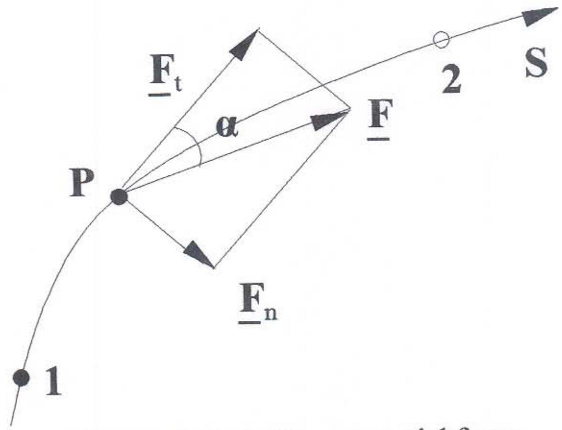


Figure 6. Only the tangential force component does work.

The integral involving the force represents the work done while the integral involving the velocity represents the energy expended by the particle. When the *kinetic energy*, T, is defined as ½ mv^2, Equation (13.2-2) becomes

$$U_{1\text{-}2} = T_2 - T_1 \quad or \quad T_1 + U_{1\text{-}2} = T_2 \; . \qquad \textbf{(13.2-3)}$$

Equation (13.2-3) describes the principle of work and energy demonstrating that the work done by a force is equal to the change in the kinetic energy of the particle. Since the kinetic energy is always positive regardless of the direction that the particle moves, it represents the capacity to do work associated with the speed of the particle. The units of kinetic energy are the same as work; namely, ft·lb (U.S.) or N·m (MKS). Sometimes one introduces a unit called a Joule (J) where 1 J = 1 N·m.

13.3 Applications of the Principle of Work and Energy

The method of work and energy greatly simplifies the solution to problems involving forces, displacements, and velocities. The advantages of the methods are that there is no need to determine $\underline{a}$ and integrate to get the velocity and the position; the quantities in Equation (13.2-3) are scalars and there is no need to define a coordinate system; and, forces that do no work are eliminated from the solution. The disadvantages are that the method cannot be used to directly determine $\underline{a}$, and it cannot be used to determine a force directed normal to the path of a particle.

13.4 Systems of Particles

The method of work and energy can be applied to each particle or formulated for the system as a whole. In this case, T would represent the sum of the kinetic energies and U would be the work done by both the internal and external forces. For systems involving inextensible cords or links, pairs of internal forces do equal but opposite work and only the external forces need be considered.

13.5 Potential Energy

The concept of potential energy can be used when the elementary work, dU, of a force is an exact differential. It that case, it is possible to establish a function, V, called the potential energy, such that,

$$dU = - dV \quad . \tag{13.5-1}$$

By integrating the terms on both sides of Equation (13.5-1),

$$\boxed{U_{1-2} = V_1 - V_2 = - (V_2 - V_1) \quad .} \tag{13.5-2}$$

Since the force must be conservative, work is independent of the path and equal to the change in potential energy.

Recall that for a particle having a weight, W,

$$U_{1-2} = - W (y_2 - y_1) \quad . \tag{13.5-3}$$

By comparing the terms on the right hand side of Equations (13.5-2) and (13.5-3),

$$\boxed{V = W y \quad .} \tag{13.5-4}$$

Thus, V provides a measure of the work that may be done by particle due to its weight. Since one is interested in the relative change in potential, the choice of the datum point is inconsequential.

For the case of a spring,

$$U_{1-2} = - \frac{k}{2} (x_2^2 - x_1^2) \quad . \tag{13.5-5}$$

By comparing the terms on the right hand side of Equations (13.5-2) and (13.5-5), By comparing the terms on the right hand side of Equations (13.5-2) and (13.5-5),

$$V = \frac{1}{2} k x^2 . \qquad \textbf{(13.5-6)}$$

Example: See Sample Problem 13.6 on pages 761 of the 6th edition of Beer and Johnson.

13.6 Conservation of Energy

The energy of any system is constant and for a particle, or a system of particles, moving under the action of conservative forces,

$$U_{1\text{-}2} = T_2 - T_1 = V_1 - V_2 \qquad \textbf{(13.6-1)}$$

or

$$T_1 + V_1 = T_2 + V_2 . \qquad \textbf{(13.6-2)}$$

Equation (13.6-2) reveals that the sum of the kinetic and potential energy of a system remains constant; the quantity (T + V) represents the total mechanical energy E.

It was demonstrated in Chapter 12, for example, that when a particle P moves under a central force F, the angular momentum H_O of the particle about the center of force 0 is constant. If the force F is also conservative, there exists a potential energy, V, associated with F, and according to Equation (13.6-2), the total energy of the particle is constant. Thus, when a particle moves under a conservative central force, both the principle of conservation of angular momentum and the principle of conservation of energy can be used to study its motion.

It must be reinforced that Equation (13.6-2) holds only for a conservative force system. A friction force, for example, is an example of a non-conservative force. The work of a friction force is always negative and when a mechanical system involves friction, the energy decreases since some energy is transformed into heat (thermal energy).

13.7 Power and Efficiency

The *power*, P, is defined as the time rate at which work is done, and may be expressed as

$$P = \frac{dU}{dt} \quad . \tag{13.7-1}$$

For the particle depicted in Figure 1, dU = F cos α ds, and v = ds/dt; consequently,

$$\boxed{P = \frac{dU}{ds} \cdot \frac{ds}{dt} = (F \cos \alpha)\, v \quad .} \tag{13.7-2}$$

Power is usually expressed in terms of horsepower (HP) in the U.S. system where 1 HP = 550 ft·lb/s = 6600 in·lb/s. In the MKS system, the power is expressed in watts (W) where 1 W = 1 J/s = 1 N·m/s. The conversion between units is

$$\boxed{1\ HP = 746\ W \quad .} \tag{13.7-3}$$

The efficiency of a system, η, is defined as

$$\boxed{\eta = \frac{power\ in}{power\ out} \quad .} \tag{13.7-4}$$

For non-conservative systems such as those that include friction, $\eta < 1$.

13.8 Impulse and Momentum

Linear momentum was discussed in Section 12.5 and an alternative approach to Newton's second law and kinematics developed. Recall

$$\boxed{\sum \underline{F} = \frac{d}{dt}(m\ \underline{v})} \tag{13.8-1}$$

where the linear momentum, m$\underline{v}$, is expressed in terms of lb·s (U.S.) or N·s (MKS).

Separating variables and integrating Equation (13-8.1),

$$\int_{t_1}^{t_2} \underline{F}\, dt = m \int_{v_1}^{v_2} d\underline{v} \qquad \textbf{(13.8-2)}$$

and

$$\boxed{\int_{t_1}^{t_2} \underline{F}\, dt = m\, \underline{v}_2 - m\, \underline{v}_1 \ .} \qquad \textbf{(13.8-3)}$$

For plane motion, Equation (13.8-3) can be expressed in terms of scalar equations along the two Cartesian axes as

$$\int_{t_1}^{t_2} F_x\, dt = m\, v_{x_2} - m\, v_{x_1} \qquad \textbf{(13.8-4)}$$

and

$$\int_{t_1}^{t_2} F_y\, dt = m\, v_{y_2} - m\, v_{y_1} \ . \qquad \textbf{(13.8-5)}$$

The terms on the left hand side of equations (13.8-4) and (13.8-5) represent the areas under F_x versus t and F_y versus t curves, respectively.

If the linear impulse is defined as

$$\boxed{Imp_{1\text{-}2} = \int_{t_1}^{t_2} \underline{F}\, dt} \qquad \textbf{(13.8-6)}$$

then Equation (13.8-3) can be rewritten as

$$Imp_{1-2} = m\,\underline{v}_2 - m\,\underline{v}_1 \quad . \tag{13.8-7}$$

In the case where several forces act on a particle

$$\boxed{\sum Imp_{1-2} = m\,\underline{v}_2 - m\,\underline{v}_1 \quad .} \tag{13.8-8}$$

Equation (13.8-8) describes the principle of impulse and momentum and can be expressed in scalar form when dealing with plane motion.

13.9 Impulsive Motion

Consider a large force acting over a short period of time which produces a change in momentum. The force is called an impulsive force; the resulting motion is impulsive. In this case, the linear impulse is simply $\underline{F}\Delta t$ and Equation (13.8-8) becomes

$$\boxed{\underline{F}\,\Delta t = m\,\underline{v}_2 - m\,\underline{v}_1 \quad .} \tag{13.9-1}$$

Non-impulsive forces include the weight of a body, forces exerted by a spring, or any force known to be small compared to the impulsive forces. Unknown reactions may or may not be impulsive and should be included when applying Equation (13.9-1).

13.10 Systems of Particles

Equation (13.8-8) holds true for a particular particle. For a system of particles, the internal forces act in pairs, and

$$\boxed{\sum External\ Imp_{1-2} = \sum m\,\underline{v}_2 - \sum m\,\underline{v}_1 \quad .} \tag{13.10-1}$$

Equation (13.10-1) can be modified for plane motion by expressing it in scalar form and considering the equations for the mass center. The latter are given by

$$\sum m\,\bar{x} = \sum m\,x \qquad and \qquad \sum m\,\bar{y} = \sum m\,y \quad . \tag{13.10-2}$$

Differentiating the terms in Equation (13.10-2),

$$\sum m\,\bar{v}_x = \sum m\,v_x \qquad and \qquad \sum m\,\bar{v}_y = \sum m\,v_y \ . \tag{13.10-3}$$

Incorporating the relations included in Equation (13.10-3) into the scalar form of Equation (13.10-1) gives,

$$\begin{aligned} \int_{t_1}^{t_2} F_{x_{external}}\,dt &= \sum m\,\bar{v}_{x_2} - \sum m\,\bar{v}_{x_1} \\ \int_{t_1}^{t_2} F_{y_{external}}\,dt &= \sum m\,\bar{v}_{y_2} - \sum m\,\bar{v}_{y_1} \ . \end{aligned} \tag{13.10-4}$$

The relations in Equation (13.10-4) verify that the mass center moves as if the entire mass and all of the external forces were concentrated at that point.

13.11 Conservation of Momentum

When the sum of the impulses of the external forces is zero, Equation (13.10-1) becomes

$$\sum m\,\underline{v}_1 = \sum m\,\underline{v}_2 \ . \tag{13.11-1}$$

Introducing the mass center by using the expressions in Equation (13.10-3),

$$\bar{\underline{v}}_1 = \bar{\underline{v}}_2 \ . \tag{13.11-2}$$

Equation (13.11-2) shows that the mass center of the system moves with a constant velocity.

13.12 Direct Central Impact

Figure 7 shows a collision taking place between two deformable bodies.

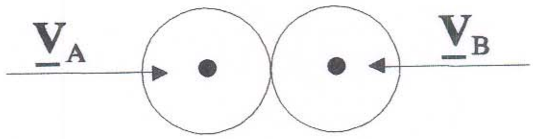


Figure 7. Direct central impact.

The line that is perpendicular to the plane of contacting surfaces is called the line of impact. *Direct central impact* occurs when the mass centers of two bodies are on the line of impact when the collision takes place.

Consider, for argument sake, two particles A and B moving along the same straight line to the right with some velocity. If the velocity of A is greater than that of B, A hits B, deformation occurs, and restitution takes place to an extent that depends upon the impact forces and the materials involved.

Since the total momentum is conserved

$$m_A \underline{v}_A + m_B \underline{v}_B = m_A \underline{v}_A' + m_B \underline{v}_B' \quad . \tag{13.12-1}$$

Equation (13.12-1) is really a vector equation but, in the case of direct central impact, all of the velocities lie along the same line and the equation becomes

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \quad . \tag{13.12-2}$$

The relative velocities of the two particles after impact is related to those before impact through the coefficient of restitution, e, as follows:

$$e = \frac{(v_B' - v_A')}{(v_A - v_B)} \quad . \tag{13.12-3}$$

Equations (13.12-2) and (13.12-3) can be used to determine the final velocities of particles initially impacting with velocities v_A and v_B.

The value of e lies in the range $0 \le e \le 1$ but two cases are of special interest. When $e = 0$ the impact is *perfectly plastic*. In this case, Equation (13.12-3) predicts that

$$v_B' = v_A' = v' \quad . \tag{13.12-4}$$

Substituting Equation (13.12-4) into Equation (13.12-2)

$$m_A v_A + m_B v_B = (m_A + m_B)\, v' \quad . \tag{13.12-5}$$

When e = 1, the impact is classified as *perfectly elastic*. In this case, the kinetic energy is conserved and

$$\frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A v_A'^2 + \frac{1}{2} m_B v_B'^2 \quad . \tag{13.12-6}$$

In general, mechanical energy is not conserved during impact, since it is turned into heat and elastic waves within the two colliding bodies.

13.13 Problems Involving Energy and Momentum

There is no real advantage in using impulse and momentum to solve problems involving no impulsive forces, however, the method provides the only practical approach to solve impact problems.

Many problems involve conservative forces except for a short phase during which impulsive forces act. A combination of methods can be used in these problems (Newton's second law, work and energy, impulse and momentum).

Example:

The 2-kg block, labeled A in Figure 8, is moving with a velocity $\underline{v}_A$, of magnitude 4 m/s, when it hits the 1.5-kg sphere labeled B. The sphere is hanging from a cord at O and is at rest when it is struck. If $\mu_k = 0.6$ between the block and the horizontal surface and e = 0.8 between the block and the sphere, determine

(a) the maximum height, h, reached by the sphere after impact.
(b) the distance, d, traversed by the block before it comes to rest.

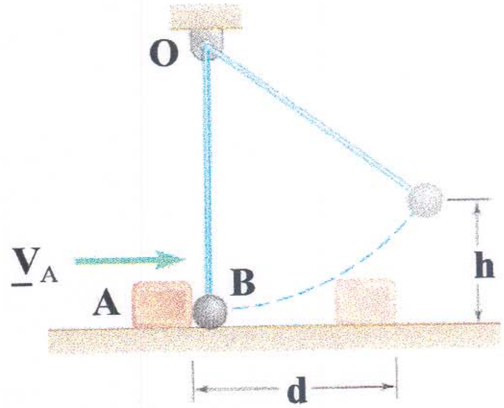


Figure 8. A block hits a sphere.

Noting that linear momentum is conserved and applying Equation (13.12-2) along the x direction, positive to the right,

$$m_A\, v_A + m_B\, v_B = m_A\, v_A' + m_B\, v_B'$$

$$2\,(4) + 1.5\,(0) = 2\, v_A' + 1.5\, v_B' \qquad or \qquad 2\, v_A' + 1.5\, v_B' = 8 \quad . \tag{1}$$

The relative velocities are related through Equation (13.12-3) as follows:

$$e = \frac{(v_B' - v_A')}{(v_A - v_B)}$$

$$0.8 = \frac{(v_B' - v_A')}{(4 - 0)} \qquad or \qquad v_B' = 3.2 + v_A' \quad . \tag{2}$$

Substituting Equation (2) into Equation (1) to obtain v_A', and then applying Equation (2) again,

$$v_A' = 0.914\ \frac{m}{s} \qquad and \qquad v_B' = 4.11\ \frac{m}{s} \quad . \tag{3}$$

Figure 9 shows the sphere with the datum at ground level. Applying conservation of energy,

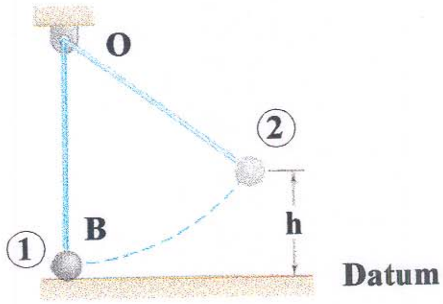


Figure 9. Consider the sphere.

$$T_1 + V_1 = T_2 + V_2 \qquad or \qquad \frac{1}{2}\, m_B\, v_B'^{\,2} + m_B\, g\,(0) = \frac{1}{2}\, m_B\,(0)^2 + m_B\, g\, h \tag{4}$$

$$\frac{1}{2}\,(1.5)\,(4.11)^2 = 1.5\,(9.81)\,h \ .$$

Solving Equation (4) for h,

$$\boxed{h = 0.86\ m = 2.8\ ft \ .} \tag{5}$$

Figure 10, on the other hand, shows the free body diagram of the block.

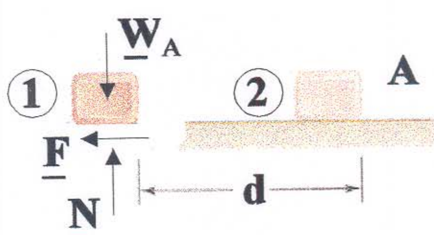


Figure 10. Consider the block.

Since the block moves horizontally,

$$+\!\uparrow \sum F_y = m_A\, a_y \qquad or \qquad N - W_A = 0$$

$$N = W_A = m_A\, g \qquad and \qquad F = \mu_k\, N = \mu_k\, m_A\, g \ . \tag{6}$$

Applying the principle of work and energy to the block and using Equation (6),

$$U_{1\to 2} = T_2 - T_1 \qquad or \qquad - F\, d = \frac{1}{2}\, m_A\, (0) - \frac{1}{2}\, m_A\, v_A'^{\,2}$$

$$0.6\ (2)\ (9.81)\ d = \frac{1}{2}\ (2)\ (0.914)^2 \tag{7}$$

Solving Equation (7) for d,

$$\boxed{d = 0.07\ m = 0.23\ ft\ .} \tag{8}$$

Chapter 14. Work and Energy

14.1 Newton's Second Law Applied to a System of Particles

Figure 1 shows two particles P_i and P_j having mass m_i and m_j, respectively. The forces $\underline{f}_{ij}$ and $\underline{f}_{ji}$ are mutual central forces; and, $\underline{F}_i$ and $\underline{F}_j$ are external forces.

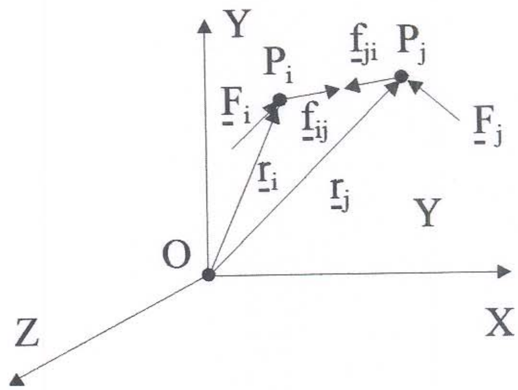


Figure 1. Interaction between two particles.

Applying Newton's second law to both particles,

$$\begin{aligned} \underline{F}_i + \underline{f}_{ij} &= m_i\, \underline{a}_i \qquad \textit{for } P_i \\ \underline{F}_j + \underline{f}_{ji} &= m_j\, \underline{a}_j \qquad \textit{for } P_j \; . \end{aligned} \qquad \textbf{(14.1-1)}$$

The analysis can be extended to three forces,

$$\underline{F}_i + \underline{f}_{ij} + \underline{f}_{ik} = m_i\, \underline{a}_i \qquad \textit{for } P_i\textit{, etc.} \qquad \textbf{(14.1-2)}$$

and when the approach is generalized to n particles,

$$\underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij} = m_i\, \underline{a}_i \qquad for\ P_i$$
$$\underline{F}_j + \sum_{i=1}^{n} \underline{f}_{ji} = m_j\, \underline{a}_j \qquad for\ P_j,\ etc. \tag{14.1-3}$$

In Equation (14.1-3) the terms $\underline{f}_{ii}$ and $\underline{f}_{jj}$ have no meaning.

As opposed to the individual particles, Newton's second law can be applied to the entire system,

$$\sum_{i=1}^{n} \underline{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{f}_{ij} = \sum_{i=1}^{n} m_i\, \underline{a}_i \ . \tag{14.1-4}$$

The second term in Equation (14.1-4) is zero, since the mutual central forces occur in pairs, and

$$\boxed{\sum_{i=1}^{n} \underline{F}_i = \sum_{i=1}^{n} m_i\, \underline{a}_i \ .} \tag{14.1-5}$$

14.2 Moments of Forces On a System of Particles

Referring again to Figure 1, consider taking moments about point O. For particle P_i,

$$\underline{r}_i \times \underline{F}_i + \underline{r}_i \times \underline{f}_{ij} = \underline{r}_i \times m_i\, \underline{a}_i \ . \tag{14.2-1}$$

For a system of particles,

$$\sum_{i=1}^{n} (\underline{r}_i \times \underline{F}_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} (\underline{r}_i \times \underline{f}_{ij}) = \sum_{i=1}^{n} (\underline{r}_i \times m_i\, \underline{a}_i) \ . \tag{14.2-2}$$

The second term on the left hand side of Equation (14.2-2) is zero. Thus, the governing equation for the motion of the system about the point O, caused by the moment of the forces, is

$$\sum_{i=1}^{n} (\underline{r}_i \times \underline{F}_i) = \sum_{i=1}^{n} \underline{M}_{iO} = \sum_{i=1}^{n} (\underline{r}_i \times m_i\, \underline{a}_i) \quad . \tag{14.2-3}$$

14.3 Motion of the Center of Mass

For a complete description of the motion of a system, it is useful to determine the position, velocity, and acceleration of the center of mass (center of gravity in U.S. jargon), G. By definition

$$m\, \underline{r}_G = \sum_{i=1}^{n} m_i\, \underline{r}_i \tag{14.3-1}$$

where $\underline{r}_i$ is the position vector to each individual particle, m_i is the mass of an individual particle, $\underline{r}_G$ is the position vector to the center of mass, and m is the total mass of the system.

The velocity and acceleration of the center of mass can be determined by differentiating Equation (14.3-1) as

$$m\, \underline{v}_G = \sum_{i=1}^{n} m_i\, \underline{v}_i \tag{14.3-2}$$

and

$$m\, \underline{a}_G = \sum_{i=1}^{n} m_i\, \underline{a}_i \quad . \tag{14.3-3}$$

Substituting Equation (14.3-3) into Equation (14.1-5),

$$\sum_{i=1}^{n} \underline{F}_i = m\, \underline{a}_G \quad . \tag{14.3-4}$$

It is apparent from Equation (14.3-4) that if $\Sigma\, \underline{F}_i = 0$, $\underline{a}_G = 0$, and G remains at rest or moves with a constant velocity. In general, the resultant force does not act through G, causing both translation and rotation. Equation (14.3-4) describes only the motion pertaining to the translation.

14.4 Linear and Angular Momentum Of a System of Particles

By definition, the linear momentum of a system of particles is

$$\underline{L} = \sum_{i=1}^{n} m_i \, \underline{v}_i \quad . \tag{14.4-1}$$

But from Equation (14.3-2)

$$\underline{L} = m \, \underline{v}_G \tag{14.4-2}$$

Differentiating Equation (14-4.2) while assuming that the mass remains constant,

$$\dot{\underline{L}} = m \, \underline{a}_G \tag{14.4-3}$$

or

$$\boxed{\sum_{i=1}^{n} \underline{F}_i = \dot{\underline{L}} \quad .} \tag{14.4-4}$$

Thus, the resultant of all external forces on a system of particles equals the time rate of change of the linear momentum of that system.

By definition, the angular momentum of a system of n particles with respect to an arbitrary fixed reference point is

$$\underline{H}_O = \sum_{i=1}^{n} (\underline{r}_i \; x \; m_i \, \underline{v}_i) \quad . \tag{14.4-5}$$

Differentiating equation (14.4-5),

$$\underline{\dot{H}}_O = \sum_{i=1}^{n} [(\underline{\dot{r}}_i \times m_i \underline{v}_i) + (\underline{r}_i \times m_i \underline{\dot{v}}_i)] = \sum_{i=1}^{n} (\underline{r}_i \times m_i \underline{a}_i) \quad . \tag{14.4-6}$$

The first term contained between the square brackets in Equation (14.4-6) is zero, since the vectors are collinear; and, the right hand side of the equation represents the sum of the moments about the origin. Therefore,

$$\sum_{i=1}^{n} \underline{M}_O = \underline{\dot{H}}_O \quad . \tag{14.4-7}$$

Thus, the resultant of the moments of the external forces on a system of particles equals the time rate of change of the angular momentum of the system.

14.5 Angular Momentum About The Center of Mass

It is advantageous to compute the angular momentum, $\underline{H}_G$, about the center of mass but one must consider what effects arise if the system of particles is moving. To this end, assume that G is moving with respect to the fixed coordinate system shown in Figure 2.

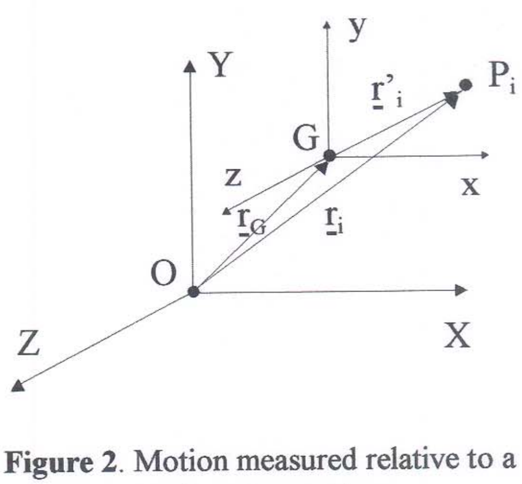


Figure 2. Motion measured relative to a moving mass center.

Referring to Figure 2, the position vectors to P_i measured relative to the two coordinate systems are related via the equation

$$\underline{r}_i = \underline{r}_G + \underline{r}_i' \ . \tag{14.5-1}$$

By successively differentiating the relation in Equation (14.5-1),

$$\underline{v}_i = \underline{v}_G + \underline{v}_i' \qquad \textit{and} \qquad \underline{a}_i = \underline{a}_G + \underline{a}_i' \ . \tag{14.5-2}$$

Now, the absolute angular momentum of the system of particles is defined with respect to the origin of the (X,Y,Z) system as

$$\underline{H}_O = \sum_{i=1}^{n} (\underline{r}_i \ x \ m_i \ \underline{v}_i) \tag{14.5-3}$$

whereas the angular momentum of the particles that move at velocities v_i' with respect to G is

$$\underline{H}_G = \sum_{i=1}^{n} (\underline{r}_i' \ x \ m_i \ \underline{v}_i') \ . \tag{14.5-4}$$

By introducing the first relation contained in Equation (14.5-2) into Equation (14.5-4),

$$\underline{H}_G = \sum_{i=1}^{n} (\underline{r}_i' \ x \ m_i \ \underline{v}_i) - \sum_{i=1}^{n} (\underline{r}_i' \ x \ m_i \ \underline{v}_G) \ . \tag{14.5-5}$$

But

$$\sum_{i=1}^{n} (\underline{r}_i' \ x \ m_i \ \underline{v}_G) = (\sum_{i=1}^{n} m_i \ \underline{r}_i') \ x \ \underline{v}_G \tag{14.5-6}$$

and, since G is at the origin of the (x,y,z) system,

$$\sum_{i=1}^{n} m_i \underline{r}_i' = 0 \quad . \tag{14.5-7}$$

Therefore,

$$\boxed{\underline{H}_G = \sum_{i=1}^{n} (\underline{r}_i' \times m_i \underline{v}_i') = \sum_{i=1}^{n} (\underline{r}_i' \times m_i \underline{v}_i) \quad .} \tag{14.5-8}$$

Thus, the angular momentum of a system of particles about its center of mass is the same whether it is observed from a (fixed) Newtonian reference frame or from a centroidal frame which may be translating but not rotating. This result is not true for points other than the center of mass.

Now, by using the second term in Equation (14.5-8) and differentiating the relation,

$$\dot{\underline{H}}_G = \sum_{i=1}^{n} (\dot{\underline{r}}_i' \times m_i \underline{v}_i') + \sum_{i=1}^{n} (\underline{r}_i' \times m_i \dot{\underline{v}}_i') \quad . \tag{14.5-9}$$

The first term on the right hand side of Equation (14.5-9) is zero, and by rearranging terms in the second expression contained in Equation (14.5-2),

$$\underline{a}_i' = \underline{a}_i - \underline{a}_G \quad . \tag{14.5-10}$$

By eliminating the zero term in Equation (14.5-9) and substituting the expression in Equation (14.5-10),

$$\dot{\underline{H}}_G = \sum_{i=1}^{n} [\underline{r}_i' \times m_i (\underline{a}_i - \underline{a}_G)] \quad . \tag{14.5-11}$$

Equation (14.5-11) can be expanded as

$$\dot{\underline{H}}_G = \sum_{i=1}^{n} (\underline{r}_i' \times m_i \underline{a}_i) - (\sum_{i=1}^{n} m_i \underline{r}_i') \times \underline{a}_G \quad . \tag{14.5-12}$$

The summation in the right hand term is zero and by introducing Newton's second law, the equation can be written as

$$\dot{\underline{H}}_G = \sum_{i=1}^{n} (\underline{r}_i' \times \underline{F}_i) = \sum_{i=1}^{n} \underline{M}_{iG} \quad . \qquad \textbf{(14.5-13)}$$

Thus,

$$\boxed{\sum_{i=1}^{n} \underline{M}_G = \dot{\underline{H}}_G \quad .} \qquad \textbf{(14.5-14)}$$

Equation (14.5-14) shows that the resultant of the moments of the external forces about the center of mass of a system of particles equals the time rate of change of the angular momentum about the center of mass.

The relationship between $\underline{H}_O$ and $\underline{H}_G$ can be established as follows:

$$\begin{aligned} \underline{H}_O &= \sum_{i=1}^{n} (\underline{r}_i \times m_i \, \underline{v}_i) = \sum_{i=1}^{n} [(\underline{r}_G + \underline{r}_i') \times m_i \, \underline{v}_i] \\ &= \sum_{i=1}^{n} (\underline{r}_G \times m_i \, \underline{v}_i) + \sum_{i=1}^{n} (\underline{r}_i' \times m_i \, \underline{v}_i) \\ &= \underline{r}_G \times \sum_{i=1}^{n} m_i \, \underline{v}_i + \sum_{i=1}^{n} (\underline{r}_i' \times m_i \, \underline{v}_i) \quad . \end{aligned} \qquad \textbf{(14.5-15)}$$

Introducing Equations (14.2-2) and (14.5-8),

$$\boxed{\underline{H}_O = \underline{r}_G \times m \, \underline{v}_G + \underline{H}_G \quad .} \qquad \textbf{(14.5-16)}$$

Also,

$$\sum_{i=1}^{n} \underline{M}_O = \dot{\underline{H}}_O = \frac{d}{dt} [\underline{H}_G + (\underline{r}_G \times m \, \underline{v}_G)] = \dot{\underline{H}}_G + \dot{\underline{r}}_G \times m \, \underline{v}_G + \underline{r}_G \times m \, \dot{\underline{v}}_G \quad . \qquad \textbf{(14.5-17)}$$

Since the second term on the right hand side is zero, Equation (14.5-17) becomes

$$\sum_{i=1}^{n} \underline{M}_O = \dot{\underline{H}}_G + \underline{r}_G \times m\, \underline{a}_G \quad . \tag{14.5-18}$$

Example: Figure 3 shows a system consisting of three particles A, B, and C. The masses of the particles are $m_A = 1$ kg, $m_B = 2$ kg, and $m_C = 3$ kg; their velocities, expressed in m/s are, respectively, $\underline{v}_A = 3\underline{i} - 2\underline{j} + 4\underline{k}$, $\underline{v}_B = 4\underline{i} + 3\underline{j}$, and $\underline{v}_C = 2\underline{i} + 5\underline{j} - 3\underline{k}$. Determine (a) the position vector to the center of mass; (b) the linear momentum of the system; (c) the angular momentum of the system about the mass center; and, (d) the angular momentum about the origin. Verify Equation (14.5-16) with these results.

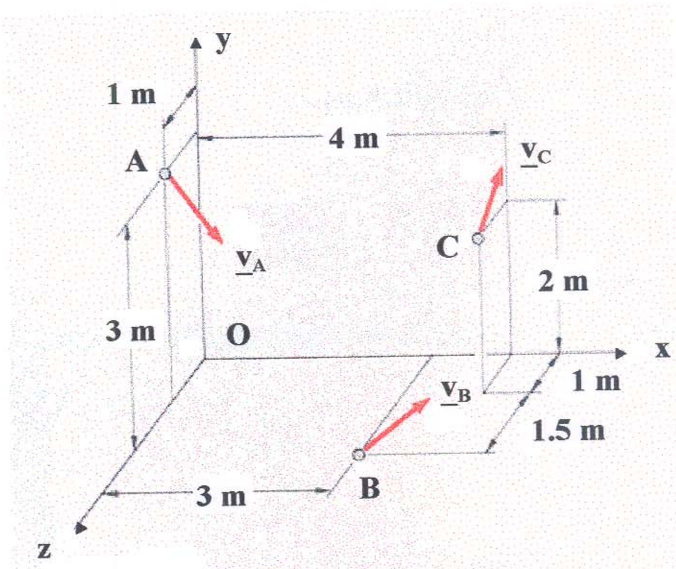


Figure 3. A system consists of three moving particles.

The position vector from the origin, O, to the center of mass, G, is determined from Equation (14.3-1) as

$$(m_A + m_B + m_C)\, \underline{r}_G = m_A\, \underline{r}_A + m_B\, \underline{r}_B + m_C\, \underline{r}_C \tag{1}$$

$$6\, \underline{r}_G = 1\,(3\,\underline{j} + \underline{k}) + 2\,(3\,\underline{i} + 2.5\,\underline{k}) + 3\,(4\,\underline{i} + 2\,\underline{j} + \underline{k}) \quad .$$

Solving Equation (1) for $\underline{r}_G$,

$$\boxed{\underline{r}_G = (3\ \underline{i} + 1.5\ \underline{j} + 1.5\ \underline{k})\ m = (9.84\ \underline{i} + 4.92\ \underline{j} + 4.92\ \underline{k})\ ft\ .} \qquad \textbf{(2)}$$

The linear momentum of the system is determined from Equation (14.4-1) as

$$\underline{L} = \sum_{i=1}^{n} m_i\ \underline{v}_i = m_A\ \underline{v}_A + m_B\ \underline{v}_B + m_C\ \underline{v}_C$$

$$= 1\ (5\ \underline{i} - 2\ \underline{j} + 4\ \underline{k}) + 2\ (4\ \underline{i} + 3\ \underline{j}) + 3\ (2\ \underline{i} + 5\ \underline{j} - 3\ \underline{k})\ . \qquad \textbf{(3)}$$

and

$$\boxed{\underline{L} = (17\ \underline{i} + 19\ \underline{j} - 5\ \underline{k})\ \frac{kg\ m}{s} = (3.82\ \underline{i} + 4.27\ \underline{j} - 1.12\ \underline{k})\ lb\ s\ .} \qquad \textbf{(4)}$$

The angular momentum about the center of mass is given by Equation (14.5-4) where the quantities $\underline{r}_i$' are computed from Equations (14.5-1) and (2) as follows:

$$\underline{r}_i' = \underline{r}_i - \underline{r}_G$$

$$\underline{r}_A' = (3\ \underline{j} + \underline{k}) - (3\ \underline{i} + 1.5\ \underline{j} + 1.5\ \underline{k}) = -3\ \underline{i} + 1.5\ \underline{j} - 0.5\ \underline{k}$$

$$\underline{r}_B' = (3\ \underline{i} + 2.5\ \underline{k}) - (3\ \underline{i} + 1.5\ \underline{j} + 1.5\ \underline{k}) = -1.5\ \underline{j} + \underline{k} \qquad \textbf{(5)}$$

$$\underline{r}_C' = (4\ \underline{i} + 2\ \underline{j} + \underline{k}) - (3\ \underline{i} + 1.5\ \underline{j} + 1.5\ \underline{k}) = \underline{i} + 0.5\ \underline{j} - 0.5\ \underline{k}\ .$$

Applying Equation (14.5-4),

$$\underline{H}_G = \sum_{i=1}^{3} (\underline{r}_i'\ x\ m_i\ \underline{v}_i) = \underline{r}_A'\ x\ m_A\ \underline{v}_A + \underline{r}_B'\ x\ m_B\ \underline{v}_B + \underline{r}_C'\ x\ m_C\ \underline{v}_C \qquad \textbf{(6)}$$

$$= (2\ \underline{i} + 24.5\ \underline{j} + 25.5\ \underline{k})\ \frac{kg\ m^2}{s} = (1.47\ \underline{i} + 18.07\ \underline{j} + 18.81\ \underline{k})\ ft\ lb\ s\ .$$

The angular momentum about the origin is given by Equation (14.4-5) as,

$$\underline{H}_O = \sum_{i=1}^{3} (\underline{r}_i \times m_i \underline{v}_i) = \underline{r}_A \times m_A \underline{v}_A + \underline{r}_B \times m_B \underline{v}_B + \underline{r}_C \times m_C \underline{v}_C$$

$$= (-34\,\underline{i} + 65\,\underline{j} + 57\,\underline{k}) \frac{kg\ m^2}{s} = (-25\,\underline{i} + 50\,\underline{j} + 42\,\underline{k})\ ft\ lb\ s \quad . \tag{7}$$

Now, applying Equation (14.5-16) while using the results contained in Equation (2), Equations (14.3-2) and (4), and Equation (6),

$$\underline{H}_O = \underline{r}_G \times m\,\underline{v}_G + \underline{H}_G$$

$$(3\,\underline{i} + 1.5\,\underline{j} + 1.5\,\underline{k}) \times (17\,\underline{i} + 19\,\underline{j} - 5\,\underline{k}) + (2\,\underline{i} + 24.5\,\underline{j} + 25.5\,\underline{k})$$

$$= (-34\,\underline{i} + 65\,\underline{j} + 57\,\underline{k}) \frac{kg\ m^2}{s} = (-25\,\underline{i} + 50\,\underline{j} + 42\,\underline{k})\ ft\ lb\ s \quad . \tag{8}$$

Thus, Equation (14.5-16) is verified by comparing the results in Equations (7) and (8).

14.6 Conservation of Momentum of A System of Particles

If the forces are zero during the period of concern, Equation (14.4-4) becomes

$$\sum_{i=1}^{n} \underline{F}_i = \dot{\underline{L}} = 0 \tag{14.6-1}$$

and

$$\underline{L} = constant \quad . \tag{14.6-2}$$

Equation (14.6-2) represents the conservation of linear momentum for the system as a whole. However, this does not mean that the linear momentum of individual particles is conserved.

Similarly, if the resultant moment of all the external forces is zero, Equation (14.4-7) becomes

$$\sum_{i=1}^{n} \underline{M}_O = \dot{\underline{H}}_O = 0 \tag{14.6-3}$$

and

$$\boxed{\underline{H}_O = constant \ .} \tag{14.6-4}$$

Similarly, Equation (14.5-14) becomes

$$\sum_{i=1}^{n} \underline{M}_G = \dot{\underline{H}}_G = 0 \tag{14.6-5}$$

and

$$\boxed{\underline{H}_G = constant \ .} \tag{14.6-6}$$

That is, the angular momentum is conserved for the entire system (not necessarily for each individual particle).

14.7 Kinetic Energy of a System of Particles

The kinetic energy of a system of particles is the sum of the kinetic energies of the individual particles of the system. Thus,

$$\boxed{T = \frac{1}{2} \sum_{i=1}^{n} m_i \, v_i^2 \ .} \tag{14.7-1}$$

In the case shown in Figure 2 where the reference frame (x,y,z) is translating with the center of mass, G, and moving at velocity $\underline{v}_G$ relative to the inertial reference system (X,Y,Z),

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} \sum_{i=1}^{n} m_i v_i'^2 \qquad \textbf{(14.7-2)}$$

where the first term on the right hand side of the equation represents the motion of the total mass imagined to be concentrated at G moving in the (X,Y,Z) frame, and the second term on the right hand side represents the motion of all particles moving at velocity, v_i', relative to the mass center at G.

14.8 Gravitational Potential Energy for a System of Particles

The gravitational potential energy of a system of particles is defined as the sum of the potential energies of the individual particles of the system. Thus,

$$V = g \sum_{i=1}^{n} m_i \underline{y}_i = \sum_{i=1}^{n} W_i \underline{y}_i \qquad \textbf{(14.8-1)}$$

where m_i and W_i are the mass and weight of particle P_i, respectively, and y is measured from an arbitrary datum. For a system whose mass center is located at $\bar{y}$ from the datum,

$$V = m g \bar{y} = W \bar{y} \ . \qquad \textbf{(14.8-2)}$$

14.9 Work and Energy - Conservation of Energy

The principle of work and energy may be applied to a system of n particles as well as to individual particles and,

$$U_{1\text{-}2} = T_2 - T_1 \qquad or \qquad T_1 + U_{1\text{-}2} = T_2 \ . \qquad \textbf{(14.9-1)}$$

If all of the forces acting on the particles of the system are *conservative*, potential energy may be introduced and

$$T_1 + V_1 = T_2 + V_2 \ . \qquad \textbf{(14.9-2)}$$

14.10 Impulse and Momentum of A System of Particles

Recall from Equation (14.4-4) that

$$\sum_{i=1}^{n} \underline{F}_i = \dot{\underline{L}} \qquad \textbf{(14.10-1)}$$

and for the case of impulsive forces,

$$\boxed{\sum_{i=1}^{n} \int_{t_1}^{t_2} \underline{F}_i \, dt = \underline{L}_2 - \underline{L}_1 = m\, \underline{v}_{G2} - m\, \underline{v}_{G1} \; .} \qquad \textbf{(14.10-2)}$$

The integral is the linear impulse and represents the summation of impulses, over all particles. Similarly,

$$\sum_{i=1}^{n} \underline{M}_{iO} = \dot{\underline{H}}_O \qquad \textbf{(14.10-3)}$$

and

$$\boxed{\sum_{i=1}^{n} \int_{t_1}^{t_2} \underline{M}_{iO} \, dt = \underline{H}_{O2} - \underline{H}_{O1} = \sum_{i=1}^{n} [\underline{r}_i \times m_i\, \underline{v}_i]_2 - \sum_{i=1}^{n} [\underline{r}_i \times m_i\, \underline{v}_i]_1 \; .} \qquad \textbf{(14.10-4)}$$

Chapter 15. Kinematics of Rigid Bodies

15.1 Introduction

This chapter is limited to the study of plane motion of a rigid body in which two types of motion may take place. A *translation* occurs when all particles forming the body move along parallel paths; translations are classified as rectilinear or curvilinear. A *rotation* occurs when the particles move in concentric circles; the rotation center may be located on or outside the body.

In the case of a translation, fixed lines in the body retain their angular orientation while in a rotation, the orientation of fixed lines change. *A general plane motion* is any plane motion which is neither a translation nor a rotation, rather a combination of the two.

15.2 Translation

Figure 1 shows two particles A and B located on a body which is moving in rectilinear or curvilinear translation.

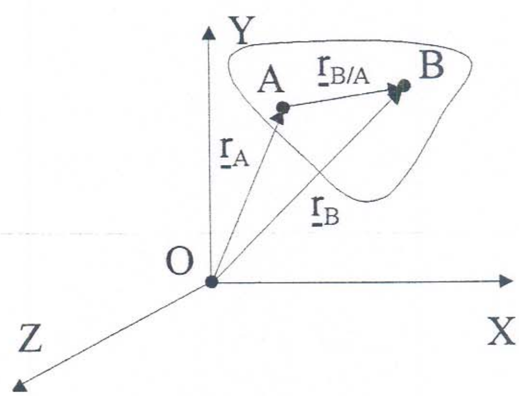


Figure 1. A body in translation.

The position vectors are related by

$$\underline{r}_B = \underline{r}_A + \underline{r}_{B/A} \quad . \qquad \textbf{(15.2-1)}$$

Differentiating Equation (15.2-1), noting that $\underline{r}_{B/A}$ has constant magnitude and direction,

$$\underline{v}_B = \underline{v}_A \tag{15.2-2}$$

and

$$\underline{a}_B = \underline{a}_A \ . \tag{15.2-3}$$

That is, when a rigid body is in translation, all points of the body have the same velocity and acceleration at a given instant.

15.3 Rotation About A Fixed Axis

Consider a rigid body that rotates about a fixed axis perpendicular to the plane of the body, intersecting it at 0. Let θ be the angular coordinate measured positive for counterclockwise rotation. The motion is defined when

$$\theta = \theta(t) \ . \tag{15.3-1}$$

The angle is measured in U.S. and MKS systems in either radians (rad), degrees (°), or revolutions (rev) where

$$1\ rev = 2\ \pi\ rad = 360^o \ . \tag{15.3-2}$$

The angular velocity, ω, is given as

$$\omega = \frac{d\theta}{dt} = \dot{\theta} \tag{15.3-3}$$

and the angular acceleration, α, is

$$\alpha = \frac{d\omega}{dt} = \dot{\omega} = \ddot{\theta} \quad . \tag{15.3-4}$$

From Equation (15.3-3), dt = $d\theta/\omega$ and by substituting the latter into Equation (15.3-4),

$$\alpha = \omega \frac{d\omega}{d\theta} \quad . \tag{15.3-5}$$

In both the U.S. and MKS systems, ω and α are measured in rad/s or rev/s; and, rad/s^2, respectively. These quantities are really vectors, since they have both magnitude and direction. They are directed out (counterclockwise rotation) or into (clockwise rotation) the plane of the body and, when using scalar notation, can be expressed by their magnitude and a "+" or "-" sign depending upon whether rotation is counterclockwise or clockwise, respectively. Both $\underline{\omega}$ and $\underline{\alpha}$ are free vectors and are, therefore, independent of the choice of the reference point established on a given rigid body.

As mentioned previously, motion is determined if $\theta = f(t)$ is known. However, the angular acceleration is usually specified in most applications and an integration process is necessary to find ω and θ.

A *uniform rotation* occurs when the angular acceleration is equal to zero for all values of time. In this case,

$$\alpha = \frac{d\omega}{dt} = 0 \quad . \tag{15.3-6}$$

Assuming $\omega(0) = \omega_0$ [at time t = 0, $\omega = \omega_0$],

$$\int_{\omega_0}^{\omega} d\omega = 0 \tag{15.3-7}$$

and

$$\omega - \omega_0 = 0 \quad \textit{and} \quad \omega = \omega_0 = \textit{constant} \quad . \tag{15.3-8}$$

Using Equation (15.3-3)

$$\omega = \frac{d\theta}{dt} = \omega_0 \quad . \tag{15.3-9}$$

Separating variables and integrating with the initial conditions, $\theta(0) = \theta_0$ and $\theta(t) = \theta$,

$$\boxed{\theta - \theta_0 = \int_{t_0}^{t} \omega\,(t)\; dt = \omega_0 \int_{t_0}^{t} dt \qquad \textit{and} \qquad \theta = \theta_0 + \omega_0\,(t - t_0) \quad .} \tag{15.3-10}$$

Note in Equation (15.3-10) that ω_0 can be taken outside of the integral sign because it is a constant.

A *uniformly accelerated rotation*, on the other hand, occurs when the angular acceleration is a constant. In this case,

$$\alpha = \frac{d\omega}{dt} = \alpha_0 \quad . \tag{15.3-11}$$

Thus,

$$\int_{\omega_0}^{\omega} d\omega = \int_{t_0}^{t} \alpha_0\; dt = \alpha_0 \int_{t_0}^{t} dt \tag{15.3-12}$$

and

$$\boxed{\omega - \omega_0 = \alpha_0\,(t - t_0) \qquad \textit{and} \qquad \omega = \omega_0 + \alpha_0\,(t - t_0) \quad .} \tag{15.3-13}$$

Recalling from Equation (15.3-3) that $d\theta/dt = \omega$, and using ω as given by Equation (15.3-13),

$$\boxed{\theta - \theta_0 = \int_{t_0}^{t} [\; \omega_0 + \alpha_0\,(t-t_0)\;]\; dt \quad \textit{and} \quad \theta = \theta_0 + \omega_0\,(t-t_0) + \frac{1}{2}\,\alpha_0\,(t-t_0)^2 \quad .} \tag{15.3-14}$$

Also recall that

$$\omega\,\frac{d\omega}{d\theta} = \alpha = \alpha_0 \quad . \tag{15.3-15}$$

Integrating

$$\int_{\omega_0}^{\omega} \omega \; d\omega = \int_{\theta_0}^{\theta} \alpha_0 \; d\theta = \alpha_0 \int_{\theta_0}^{\theta} d\theta \tag{15.3-16}$$

and

$$\boxed{\frac{1}{2}\,\omega^2 - \frac{1}{2}\,\omega_0^2 = \alpha_0\,(\,\theta - \theta_0\,) \qquad \textit{and} \qquad \omega^2 = \omega_0^2 + 2\,\alpha_0\,(\,\theta - \theta_0\,)\;.} \tag{15.3-17}$$

15.4 Linear and Angular Velocity, Linear and Angular Acceleration in Rotation

Figure 2 shows a body that is rotating about an axis perpendicular to the figure and passing through point O.

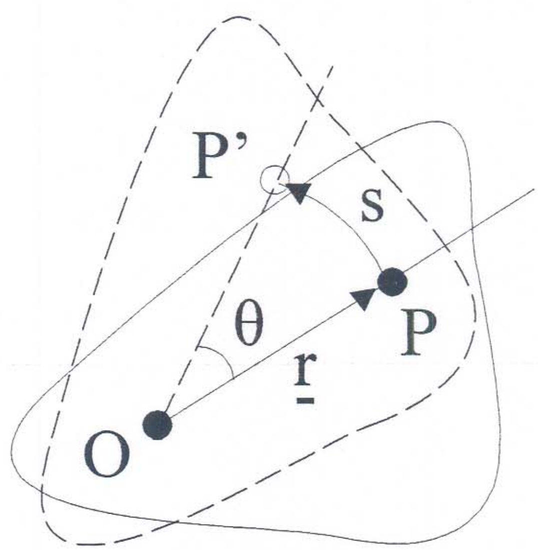


Figure 2. A rigid body rotating about O.

The arc length, s, traveled by point P, is specified in terms of the angular position coordinate, θ, by

$$\boxed{s = r\,\theta\;.} \tag{15.4-1}$$

Differentiating Equation (15.4-1),

$$\frac{ds}{dt} = r\,\frac{d\theta}{dt} \qquad \textbf{(15.4-2)}$$

and

$$\boxed{v = r\,\omega} \qquad \textbf{(15.4-3)}$$

where r is the distance between O and the point in question, v is the magnitude of the linear velocity vector, and ω is the magnitude of the angular velocity vector.

Figure 3 shows the geometrical relationship between the parameters in Equation (15.4-3) and reinforces that they are all really vector quantities.

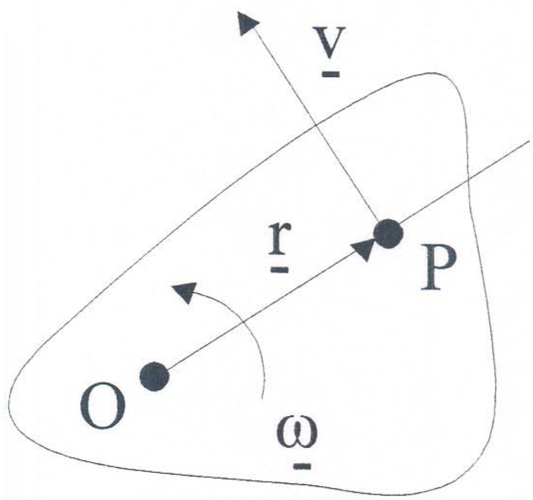


Figure 3. The relationship between parameters.

In formulating the scalar approach, the direction of the velocity vector, $\underline{v}$, is assumed to be consistent with the direction of rotation associated with the angular velocity vector, $\underline{\omega}$. It should be noted that $\underline{\omega}$ is a free vector and therefore independent of the reference point.

In vector notation, the operation in Equation (15.4-3) between the position vector and the velocity is a cross product such that

$$\boxed{\underline{v} = \frac{d\underline{r}}{dt} = \underline{\omega} \times \underline{r}\ .} \qquad \textbf{(15.4-4)}$$

Equation (15.4-3) shows that the magnitude of $\underline{v}$ depends upon that of $\underline{r}$ while it is evident from Equation (15.4-4) that, for a pure rotation, the vector $\underline{v}$ is perpendicular to $\underline{r}$. Furthermore, the direction of $\underline{v}$ is dictated by the sense of $\underline{\omega}$ (either clockwise or counterclockwise).

The acceleration can be expressed in terms of $\underline{\omega}$ and $\underline{\alpha}$ by differentiating Equation (15.4-4); namely,

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}(\underline{\omega} \times \underline{r}) = \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{\omega} \times \frac{d\underline{r}}{dt} = \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{\omega} \times \underline{v} \qquad \textbf{(15.4-5)}$$

or

$$\boxed{\underline{a} = \underline{\alpha} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})} \qquad \textbf{(15.4-6)}$$

Figure 4 shows the tangential and normal components of acceleration for the case of a pure rotation about the point O.

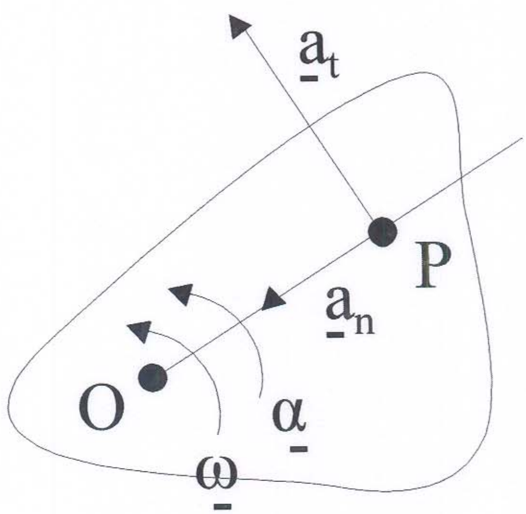


Figure 4. Tangential and normal components of $\underline{a}$ for rotation about O.

These components are given by,

$$\boxed{a_t = \frac{dv}{dt} = r\,\frac{d\omega}{dt} = r\,\alpha \qquad and \qquad a_n = \frac{v^2}{r} = \frac{r^2\,\omega^2}{r} = r\,\omega^2} \qquad \textbf{(15.4-7)}$$

In formulating the scalar approach, the tangential component of acceleration, $\underline{a}_t$, is directed consistent with the direction of $\underline{\alpha}$, while the normal component of acceleration, $\underline{a}_n$, is always directed toward the center of rotation.

15.5 General Plane Motion

A *general plane rigid body motion* may always be considered as the sum of a translation and a rotation. As illustrated in Figure 5, the motion may be replaced by a translation in which all particles of the body move along paths parallel to the path actually followed by some arbitrary reference point A, and by a rotation about the reference point A.

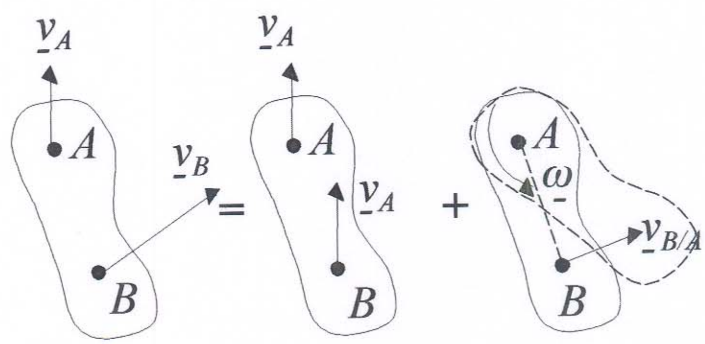


Figure 5. General plane motion.

A good example of this concept is the rolling motion depicted on page 902 of the 6th edition of Beer and Johnson. In reality, the rolling motion differs from the translation and rotation.

Referring again to Figure 5, the absolute motion of B is obtained by combining the absolute motion of A and the relative motion of B with respect to A. An observer moving with A, but not rotating, will observe B to move in a circular arc (pure rotation). The relative velocities are given by

$$\underline{v}_B = \underline{v}_A + \underline{v}_{B/A} \qquad \textbf{(15.5-1)}$$

where $\underline{v}_A$ corresponds to the translation and $\underline{v}_{B/A}$ is associated with the rotation about the reference point, A, measured with respect to a translating axes system centered at that point. From Equation (15.4-3),

$$v_{B/A} = r_{AB}\,\omega_{AB} \qquad \textbf{(15.5-2)}$$

where r_{AB} is the distance from A to B.

When formulating the argument, the angular velocity can (and should) be applied at the reference point, A, since it is a free vector. The sense of the angular velocity determines that of the relative velocity.

15.6 Instantaneous Center of Rotation In Plane Motion

At any given instant in time the velocities corresponding to a general plane motion can be looked upon as if a pure rotation was instantaneously occurring about an axis perpendicular to the body that intersects the body at a point called the instantaneous center of rotation, C.

Figure 6 illustrates the case in which $\underline{v}_A$ and $\underline{\omega}$ are known.

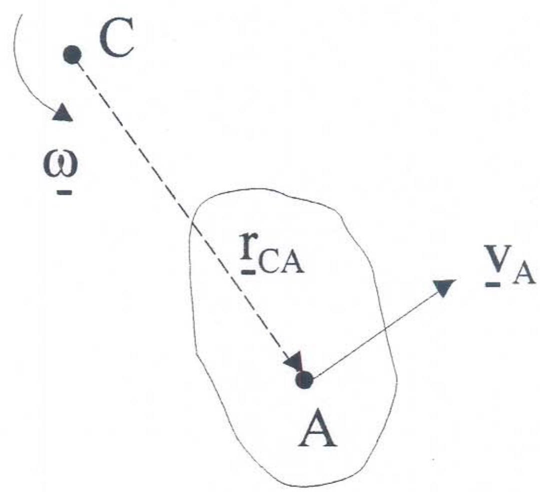


Figure 6. C is the instantaneous center.

In this case, the instantaneous center of rotation is found by moving through a distance perpendicular to $\underline{v}_A$ given by

$$r_{CA} = \frac{v_A}{\omega} . \tag{15.6-1}$$

As far as velocities are concerned, the slab seems to rotate about C.

Figure 7 illustrates the case in which the direction of the velocities of two particles are known and different. In this case, the instantaneous center is given by the intersection of the lines drawn perpendicular to the lines of action of the velocity vectors.

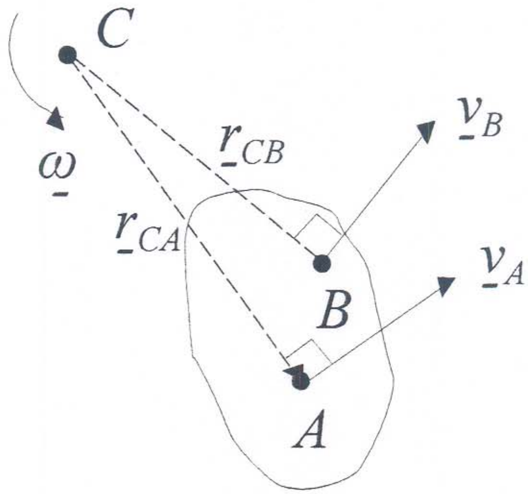


Figure 7. C is the instantaneous center.

Figure 8, on the other hand, shows the case in which the velocities are different but their lines of action are parallel. In this case, the instantaneous center is found by intersecting line AB with the line joining the extremities of the-velocity vectors. If $v_A = v_B$, the instantaneous center is at infinity and the body is in translation.

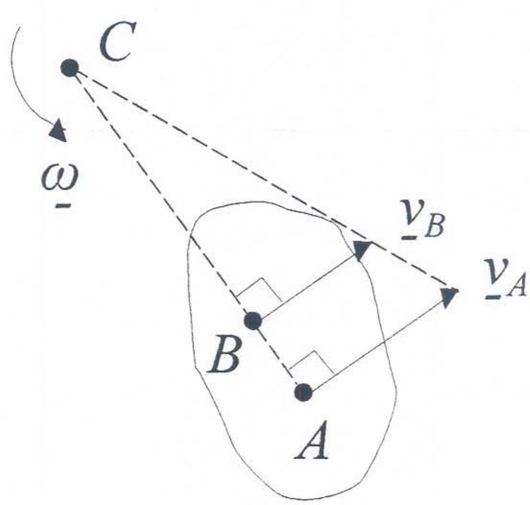


Figure 8. C is the instantaneous center.

It should be noted that this concept involves only relative velocities. If the instantaneous center,C, is on the body, $\underline{v}_C = 0$ at that instant. In general, however, C changes as a function of time and, therefore, does not have an acceleration equal to zero. Hence, ***accelerations can not be determined as if the slab were rotating about C.***

15.7 Absolute and Relative Acceleration in Plane Motion

In general,

$$\underline{a}_B = \underline{a}_A + \underline{a}_{B/A} \tag{15.7-1}$$

where $\underline{a}_A$ corresponds to the translation and $\underline{a}_{B/A}$ is associated with the rotation about the reference point, A, measured with respect to a translating axes system centered at that point. In addition,

$$\underline{a}_{B/A} = \underline{a}_{B/A_n} + \underline{a}_{B/A_t} \tag{15.7-2}$$

where these quantities are defined in Equation (15.4-7) as

$$a_{B/A_t} = \frac{dv}{dt} = r\,\frac{d\omega}{dt} = r\,\alpha \quad \textit{and} \quad a_{B/A_n} = \frac{v^2}{r} = \frac{r^2\,\omega^2}{r} = r\,\omega^2 \ . \tag{15.7-3}$$

As illustrated in Figure 9, the angular velocity and angular acceleration should be applied at the reference point. The relative velocity and tangential components of acceleration are then specified to be consistent with the chosen directions.

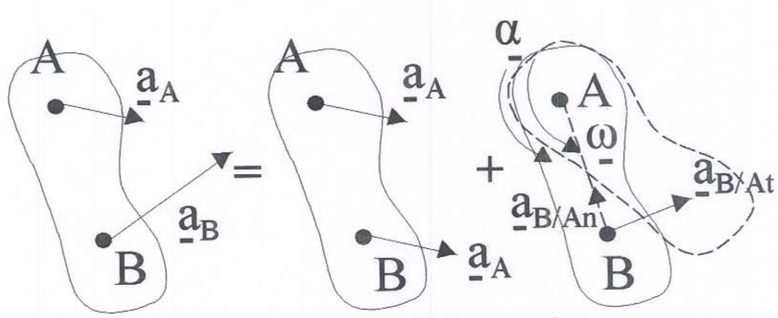


Figure 9. General plane motion.

Example: Referring to Figure 10, determine the velocity of joint B in the three-bar linkage if bar CD rotates clockwise with an angular velocity of 200 rpm.

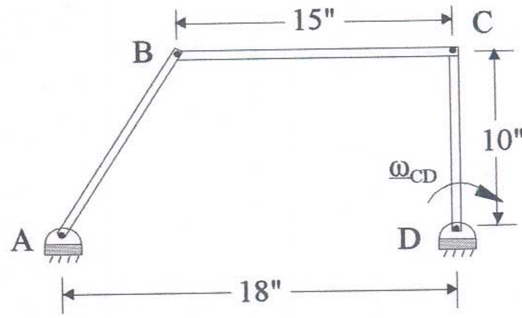


Figure 10. A mechanism on the move.

The solution to this problem can be achieved by noting that point D is fixed, and solving for the velocity of point C by considering bar CD. Then, simultaneous equations can be generated for the velocity components at B by considering bars BC and AB, separately, noting that A is also fixed. The kinematic diagrams for the bars are shown below in Figures 11 through 13.

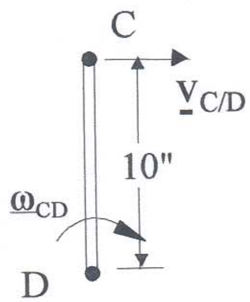


Figure 11. Bar CD.

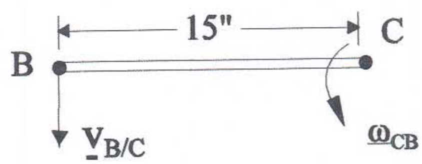


Figure 12. Bar BC.

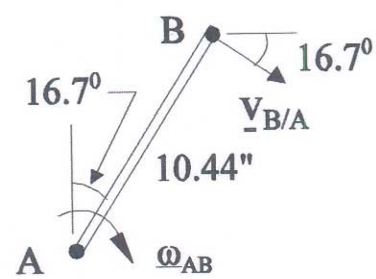


Figure 13. Bar AB.

First, note that ω_{CD} = 200 rpm = 20.94 rad/s. Then, applying Equation (15.5-1), assuming that the reference point lies at D, and referring to Figure 11,

$$\underline{v}_C = \underline{v}_D + \underline{v}_{C/D} \quad . \tag{1}$$

The two corresponding scalar equations are

$$\overset{+}{\rightarrow} \; v_{C_x} = v_{D_x} + v_{C/D_x} = 0 + r_{CD}\, \omega_{CD} = 10\ (20.94) = 209.4$$

$$+\uparrow \; v_{C_y} = v_{D_y} + v_{C/D_y} = 0 + 0 = 0 \quad . \tag{2}$$

Since the velocity is now known, a relation can be established for the velocity of point B by considering bar CB. Referring to Figure 12, assuming point C as the reference,

$$\underline{v}_B = \underline{v}_C + \underline{v}_{B/C} \quad . \tag{3}$$

The two corresponding scalar equations are

$$\overset{+}{\rightarrow}\ v_{B_x} = v_{C_x} + v_{B/C_x} = 209.4 + 0 = 209.4$$
$$+\uparrow\ v_{B_y} = v_{C_y} + v_{B/C_y} = 0 - r_{CB}\ \omega_{CB} = -\ 15\ \omega_{CB}\ . \tag{4}$$

Expressions for the velocity components of point B can also be determined by considering bar AB. Referring to Figure 13, assuming point A as the reference,

$$\underline{v}_B = \underline{v}_A + \underline{v}_{B/A}\ . \tag{5}$$

Recognizing that point A is fixed, the two corresponding scalar equations are

$$\overset{+}{\rightarrow}\ v_{B_x} = v_{A_x} + v_{B/A_x} = 0 + r_{AB}\ \omega_{AB} \cos 16.7 = 10.44\ \omega_{AB} \cos 16.7 = 10\ \omega_{AB}$$
$$+\uparrow\ v_{B_y} = v_{A_y} + v_{B/A_y} = 0 - r_{AB}\ \omega_{AB} \sin 16.7 = -\ 10.44\ \omega_{AB} \sin 16.7 = -\ 3\ \omega_{AB}\ . \tag{6}$$

Equating the right hand sides of the upper and lower expressions in Equations (4) and (6)

$$209.4 = 10\ \omega_{AB} \qquad \textit{and} \qquad -\ 15\ \omega_{CB} = -\ 3\ \omega_{AB}\ . \tag{7}$$

Solving the equations in the above for the angular velocities of the bars

$$\omega_{AB} = 20.94\ rad/s \qquad \textit{and} \qquad \omega_{CB} = 4.19\ rad/s\ . \tag{8}$$

Referring to Figure 12 and 13, the corresponding vector expressions are

$$\underline{\omega}_{AB} = 20.94\ rad/s\ \circlearrowright \qquad \textit{and} \qquad \underline{\omega}_{CB} = 4.19\ rad/s\ \circlearrowleft \tag{9}$$

Substituting the values from Equation (8) into either Equation (4) or (6),

$$\boxed{\underline{v}_B = (209.4\ \underline{i} - 62.82\ \underline{j})\ \frac{in.}{s} = (17.45\ \underline{i} - 5.24\ \underline{j})\ \frac{ft}{s} = (5.32\ \underline{i} - 1.60\ \underline{j})\ \frac{m}{s}\ .} \tag{10}$$

Example: Referring to Figure 14, the velocity and acceleration of point A are 8 ft/s and 6 ft/s^2, respectively, and both directed to the right. Determine the angular velocities and angular accelerations of bars AB and BC.

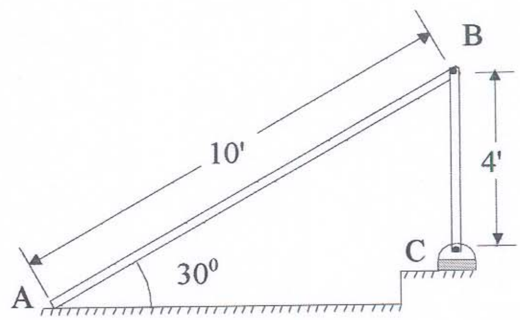


Figure 14. Point A slides along the floor.

The solution to this problem entails first finding the angular velocities of the bars using either the instantaneous center concept, or, applying the relative velocity formula to both bars and equating the expressions found for point B. Once this is accomplished, the relative acceleration formulas are applied to both bars and the expressions at B equated.

First consider the instantaneous center concept. This approach relies on the kinematic diagrams shown in Figures 15 and 16.

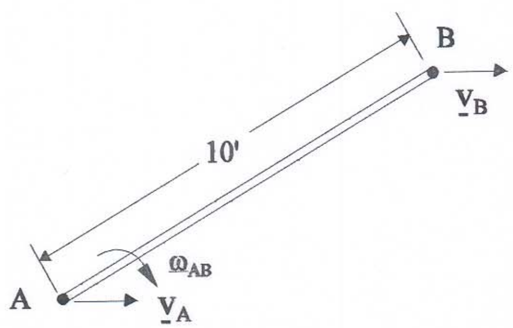


Figure 15. Bar AB.

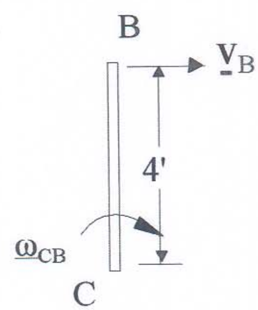


Figure 16. Bar CB.

As shown in Figure 15, the velocity of point A is horizontal and 8 ft/s directed toward the right. Since point B rotates about point C, its velocity is also horizontal and assumed directed toward the right. When lines are drawn perpendicular to the lines of actions of the velocity vectors from points A and B they never intersect, implying that the instantaneous center lies at infinity. Consequently, the bar is in translation and

$$\boxed{\omega_{AB} = 0 \ \frac{rad}{s} \ .} \qquad \textbf{(1)}$$

Since the bar is in translation, all points travel with the same velocity. Referring to Figure 16,

$$v_B = v_A = 8 \ \frac{ft}{s} \ \rightarrow \qquad and \qquad v_B = r_{CB} \ \omega_{CB} \ . \tag{2}$$

Setting $v_B = 8$ ft/s and $r_{CB} = 4$ ft

$$\boxed{\omega_{CB} = \frac{8}{4} = 2 \ \frac{rad}{s} \ \curvearrowright \ .} \tag{3}$$

An alternative approach is to apply the relative velocity formula in Equation (15.5-1) to both bars. The modified kinematic diagram for bar AB is shown below in Figure 17.

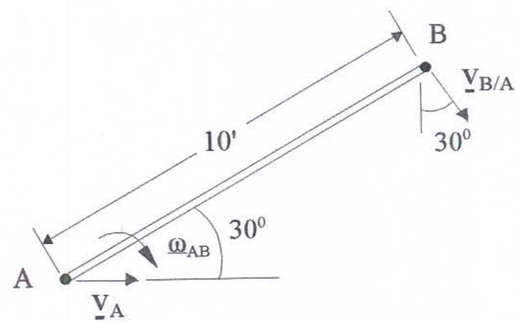


Figure 17. Bar AB.

Referring to Figures 16 and 17, respectively,

$$\begin{aligned} \underline{v}_B &= \underline{v}_C + \underline{v}_{B/C} \\ \underline{v}_B &= \underline{v}_A + \underline{v}_{B/A} \ . \end{aligned} \tag{4}$$

The scalar equations corresponding the first expression in Equation (4) and written for bar CB, are

$$\begin{aligned} \overset{+}{\rightarrow} \ v_{B_x} &= v_{C_x} + v_{B/C_x} = 0 + r_{CB} \ \omega_{CB} = 4 \ \omega_{CB} \\ +\uparrow \ v_{B_y} &= v_{C_y} + v_{B/C_y} = 0 + 0 = 0 \ . \end{aligned} \tag{5}$$

and, for the second expression in Equation (4) and bar AB,

$$\begin{aligned} \overset{+}{\rightarrow} \ v_{B_x} &= v_{A_x} + v_{B/A_x} = 8 + r_{AB} \ \omega_{AB} \sin 30 = 8 + 5 \ \omega_{AB} \\ +\uparrow \ v_{B_y} &= v_{A_y} + v_{B/A_y} = 0 - r_{AB} \ \omega_{AB} \cos 30 = -8.66 \ \omega_{AB} \ . \end{aligned} \tag{6}$$

Equating the right hand sides of the upper and lower expressions in Equations (5) and (6)

$$4\ \omega_{CB} = 8 + 5\ \omega_{AB} \qquad \textit{and} \qquad 0 = -\ 8.66\ \omega_{AB}\ . \tag{7}$$

Solving for the angular velocities

$$\boxed{\underline{\omega}_{AB} = 0 \qquad \textit{and} \qquad \underline{\omega}_{CB} = 2\ \frac{rad}{s}\ \curvearrowright\ .} \tag{8}$$

The angular accelerations can be found by considering the kinematic diagrams of the bars shown in Figures 18 and 19.

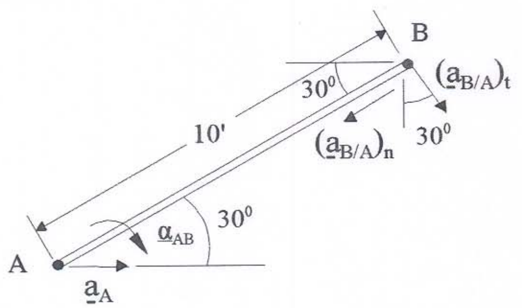


Figure 18. Bar AB.

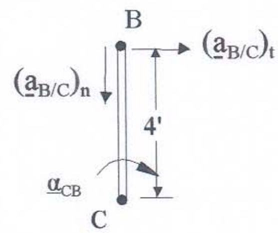


Figure 19. Bar CB.

Expressions for the acceleration of point B can be obtained by applying the relative acceleration formulas

$$\begin{aligned} \underline{a}_B &= \underline{a}_A + \underline{a}_{B/A} \\ \underline{a}_B &= \underline{a}_C + \underline{a}_{B/C}\ . \end{aligned} \tag{9}$$

Referring to Figure 18 and recognizing that $a_A = 6\ ft/s^2$ to the right, the scalar equations corresponding to the first expression in Equation (9) are

$$\begin{aligned} \overset{+}{\rightarrow}\ a_{B_x} &= a_{A_x} + a_{B/A_x} = 6 + a_{B/A_t} \sin 30 - a_{B/A_n} \cos 30 = 6 + r_{AB}\ \alpha_{AB} \sin 30 \\ &\quad - r_{AB}\ \omega_{AB}^2 \cos 30 = 6 + 10\ \alpha_{AB}\ (0.5) - 0 = 6 + 5\ \alpha_{AB} \\ +\uparrow\ a_{B_y} &= a_{A_y} + a_{B/A_y} = 0 - a_{B/A_t} \cos 30 - a_{B/A_n} \sin 30 = 0 - r_{AB}\ \alpha_{AB} \cos 30 \\ &\quad - r_{AB}\ \omega_{AB}^2 \sin 30 = 0 - 10\ \alpha_{AB}\ (0.866) - 0 = -\ 8.66\ \alpha_{AB}\ . \end{aligned} \tag{10}$$

Referring to Figure 19 and recognizing that point C is fixed, the scalar equations corresponding to the second expression in Equation (9) are

$$\overset{+}{\rightarrow}\ a_{B_x} = a_{C_x} + a_{B/C_x} = 0 + a_{B/C_t} = 0 + r_{CB}\,\alpha_{CB} = 4\,\alpha_{CB}$$

$$+\uparrow\ a_{B_y} = a_{C_y} + a_{B/C_y} = 0 - a_{B/C_n} = 0 - r_{CB}\,\omega_{CB}^{\,2} = -\,4\,(2)^2 = -\,16 \quad . \tag{11}$$

Equating the right hand sides of the upper and lower expressions in Equations (10) and (11)

$$6 + 5\,\alpha_{AB} = 4\,\alpha_{CB} \qquad and \qquad -\,8.66\,\alpha_{AB} = -\,16 \quad . \tag{12}$$

Solving the expressions in Equation (12) simultaneously, the magnitudes of the angular accelerations are

$$\boxed{\alpha_{AB} = 1.85\ \frac{rad}{s^2} \qquad and \qquad \alpha_{CB} = 3.81\ \frac{rad}{s^2} \quad .} \tag{13}$$

Or, referring to Figures 18 and 19, the corresponding vector expressions are

$$\boxed{\underline{\alpha}_{AB} = 1.85\ \frac{rad}{s^2}\ \curvearrowright \qquad and \qquad \underline{\alpha}_{CB} = 3.81\ \frac{rad}{s^2}\ \curvearrowright \quad .} \tag{14}$$

Chapter 16. Kinetics of Rigid Bodies in Plane Motion

16.1 Plane Motion of a Rigid Body - D'Alembert's Principle

This chapter considers the shape of the body and the exact location of the points of application of the forces. The ensuing study of kinetics will be limited to plane motion.

The distributed moments and forces can be reduced to a force-couple system acting at the mass center, G, of a body. This is accomplished by moving the forces to that point. The force is the resultant of the resulting concurrent force system; the moment is found by taking the moments of the original force system about G and adding these to any applied moments.

D'Alembert's principle states that the external forces acting on a body are equivalent to a force-couple system consisting of a vector of magnitude $m\underline{a}$ attached at the mass center, G, of the body, and a couple of magnitude $I_C\underline{\alpha}$ of the same sense as the angular acceleration $\underline{\alpha}$.

In this formulation, m is the mass of the body under consideration, and the quantities $\underline{a}$ and $\underline{I}$ are the linear acceleration and mass moment of inertia measured with respect to the mass center, respectively. In mathematical terms, D'Alembert's principle becomes

$$\sum \underline{F} = m\, \underline{a}_G \qquad \textbf{(16.1-1)}$$

and

$$\sum \underline{M}_C = I_G\, \underline{\alpha} \qquad \textbf{(16.1-2)}$$

Equation (16.1-1) represents the translation of the mass center while Equation (16.1-2) corresponds to a centroidal rotation.

16.2 Special Cases

In the case of a *pure translation* (no rotation), the resultant force in the force-couple system passes through the mass center and $\Sigma\, \underline{M}_G = 0$. Hence, Equation (16.1-1) fully describes the motion.

A *centroidal rotation* occurs when $\underline{a}_G = 0$ as a result of constraints imposed on G. In this case, Equation (16.1-2) fully describes the motion.

In the case of general plane motion, considered as the sum of a translation and a rotation, Equations (16.1-1) and (16.1-2) are both necessary to fully describe the motion.

16.3 Solution of Problems Involving Plane Motion of a Rigid Body - Method of Attack

The first step in solving problems involving the plane motion of a rigid body is to draw a free body diagram showing the appropriate directions chosen for $\underline{\alpha}$ and $\underline{a}_G$. A study of kinematics is performed and the appropriate equations of motion applied.

An alternative to using Equation (16.1-2) is to take moments about an arbitrary point, say O. In this case,

$$\boxed{\sum \underline{M}_O = I_O\, \underline{\alpha} = I_G\, \underline{\alpha} + \underline{r}_{OG} \times m\, \underline{a}_G\ .} \qquad \textbf{(16.3-1)}$$

Example: Figure 1 shows a 60-lb ladder that is assumed to be a uniform bar which starts sliding without frictional resistance. Determine the reactions at points A and B.

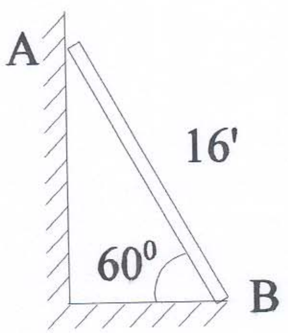


Figure 1. A ladder moves.

The solution to this problem entails a kinematic analysis to determine the linear acceleration of the center of mass of the ladder in terms of its angular acceleration. Then, a free body diagram is drawn and the three equations of motion applied to determine the angular acceleration and the two reactions.

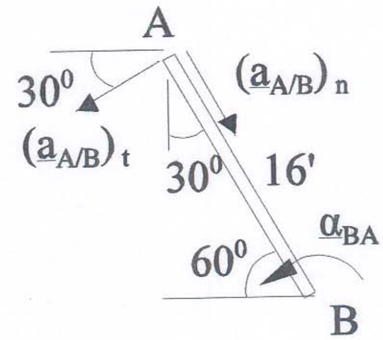


Figure 2. Kinematic diagram.

The first step in the kinematic analysis is to determine the angular velocity of the ladder so that the normal component of acceleration is known in when the relative acceleration formula is applied. This would ordinarily be done by using either the instantaneous center concept or the relative velocity formula. But, since the ladder starts from rest, the angular velocity at the instant shown is zero; that is, $\underline{\omega}_{BA} = 0$.

When the ladder begins to slip, point B moves to the right while point A moves down. The accelerations of these points are related by the relative acceleration formula

$$\underline{a}_A = \underline{a}_B + \underline{a}_{A/B} \quad . \qquad \textbf{(1)}$$

Referring to Figure 2, the corresponding scalar equations are

$$\overset{+}{\rightarrow}\ a_{A_x} = a_{B_x} + a_{A/B_x} = a_{B_x} - a_{B/A_t} \cos 30 + a_{B/A_n} \sin 30$$

$$= a_{B_x} - r_{BA}\, \alpha_{BA} \cos 30 + r_{BA}\, \omega_{BA}^{\ 2} \sin 30$$

$$+\uparrow\ a_{A_y} = a_{B_y} + a_{A/B_y} = a_{B_y} - a_{B/A_t} \sin 30 - a_{B/A_n} \cos 30 \qquad \textbf{(2)}$$

$$= a_{B_x} - r_{BA}\, \alpha_{BA} \sin 30 - r_{BA}\, \omega_{BA}^{\ 2} \cos 30 \quad .$$

Since point A moves down, the x component of its acceleration is zero and the y component is simply a_A. Similarly, since B moves to the right, the y component is zero and the x component is simply a_B. Applying this knowledge with $\underline{\omega}_{BA} = 0$, the expressions in Equation (2) become

$$\overset{+}{\rightarrow}\ 0 = a_B - 13.86\, \alpha_{BA} \qquad or \qquad a_B = 13.86\, \alpha_{BA}$$

$$+\uparrow\ a_A = -\, 8\, \alpha_{BA} \quad . \qquad \textbf{(3)}$$

The acceleration of G can now be found in terms of the angular acceleration of the ladder by applying the relative acceleration formula between point G and either point A or point B. By selecting point A as a reference, for example,

$$\underline{a}_G = \underline{a}_A + \underline{a}_{G/A} \quad . \tag{4}$$

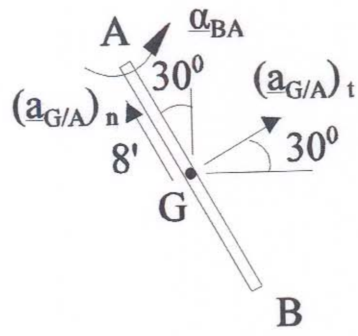


Figure 3. Center of mass.

Referring to Figure 3, the corresponding scalar equations are

$$\overset{+}{\rightarrow} \; a_{G_x} = a_{A_x} + a_{G/A_x} = a_{A_x} + a_{G/A_t} \cos 30 - a_{G/A_n} \sin 30$$

$$= a_{A_x} + r_{AG}\,\alpha_{BA} \cos 30 - r_{AG}\,\omega_{BA}^2 \sin 30 = 0 + 6.93 \;\; \alpha_{BA} - 0 = 6.93 \;\; \alpha_{BA} \tag{5}$$

$$+\uparrow \; a_{G_y} = a_{A_y} + a_{G/A_y} = a_{A_y} + a_{G/A_t} \sin 30 + a_{G/A_n} \cos 30$$

$$= a_{A_x} + r_{AG}\,\alpha_{BA} \sin 30 + r_{AG}\,\omega_{BA}^2 \cos 30 = -\,8\,\alpha_{BA} + 4\,\alpha_{BA} + 0 = -\,4\,\alpha_{BA} \quad .$$

A free body diagram of the bar is shown in Figure 4. The equations of motion are

$$\overset{+}{\rightarrow} \sum F_x = m\,a_{G_x} \qquad +\uparrow \sum F_y = m\,a_{G_y} \qquad +\circlearrowleft \sum M_G = I_G\,\alpha_{BA} \quad . \tag{6}$$

Referring to Figure 4, using Equation (5) and $I_G = 1/12\ ml^2$,

$$R_A = \frac{60}{32.2}(6.93\ \alpha_{BA}) = 12.91\ \alpha_{BA} \qquad R_B - 60 = \frac{60}{32.2}(-\ 4\ \alpha_{BA}) = -\ 7.45\ \alpha_{BA}$$

$$-\ R_A\ (8\ \sin 60) + R_B\ (8\ \cos 60) = \frac{1}{12}(16)^2\frac{60}{32.2}\alpha_{BA} \quad or \quad -\ .174\ R_A + .1\ R_B = \alpha_{BA} \tag{7}$$

Solving simultaneously,

$$\alpha_{BA} = 1.5\ \frac{rad}{s^2} \tag{8}$$

and

$$\boxed{R_A = 19.4\ lb = 86.3\ N \qquad and \qquad R_B = 48.8\ lb = 217.2\ N\ .} \tag{9}$$

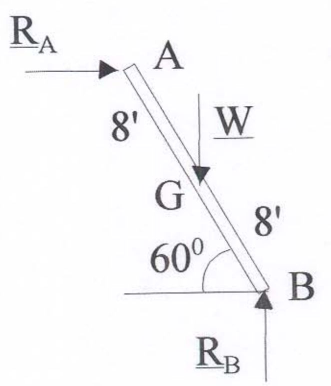


Figure 4. FBD of ladder.

Example: Figure 5 shows a uniform 10-kg wheel. It is rotating at an angular speed of $\omega = 500$ rpm when a braking force, P, of 800 N is applied to the hinged bar. The friction at O and A is negligible, while $\mu k = 0.5$ at B. Calculate the time required to stop the wheel.

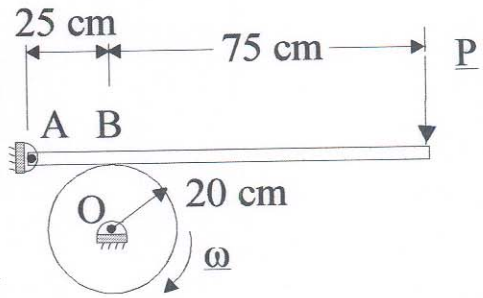


Figure 5. A brake is applied.

The solution to the problem involves applying kinetics by using the equations of motion to determine the angular acceleration. The latter is assumed to be constant and the time is found by kinematics.

The free body diagrams of the wheel and the bar are shown in Figure 6.

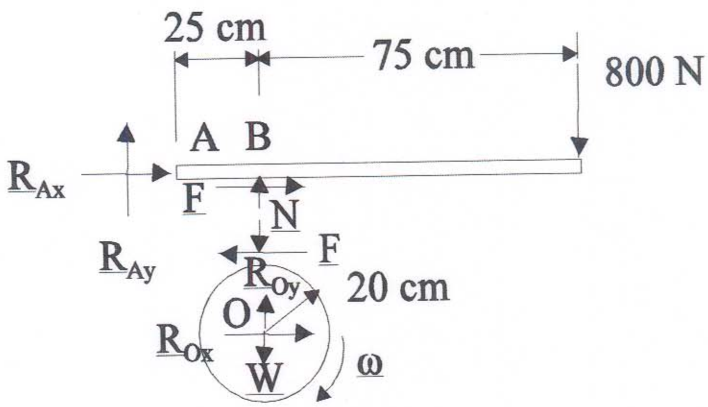


Figure 6. FBDs of the wheel and the bar.

Since the bar is in constant contact with the wheel, its angular velocity and angular acceleration are both equal to zero. Hence,

$$+\circlearrowleft \sum M_A = I_A\, \alpha_{AB} \qquad or \qquad 0.25\ N - 1.0\ (800) = 0 \tag{1}$$

$$N = 3200\ .$$

The friction force is determined by applying the kinematic relation

$$F = \mu_k\ N = 0.5\ (3200) = 1600\ . \tag{2}$$

Referring to the free body diagram of the wheel,

$$+\circlearrowleft) \quad \sum M_O = I_O\,\alpha \qquad or \qquad 0.2\,F = \frac{1}{2}\,m\,r^2\,\alpha = \frac{1}{2}\,(10)\,(0.2)^2\,\alpha \tag{3}$$

$$\alpha = 1600\,\frac{rad}{s^2}\ .$$

Assuming uniformly accelerated motion,

$$\alpha = constant \qquad and \qquad \omega = \omega_0 + \alpha\,t \tag{4}$$

Converting $\omega_0 = 500$ rpm to $\omega_0 = 52.35$ rad/s and substituting into Equation (4), recognizing that the angular velocity is negative because it is clockwise,

$$\boxed{t = -\left[\frac{-\,52.36}{1600}\right] = 0.0327\ s\ .} \tag{5}$$

Example: Rod AB is of mass mr and rod BC is of mass 2mr where m is the mass per unit length. As illustrated in Figure 7, these rods are attached to a disk as shown. A clockwise moment, M, is applied to the disk which makes it rotate in a vertical plane at constant angular velocity, ω_0. Knowing that at the position shown

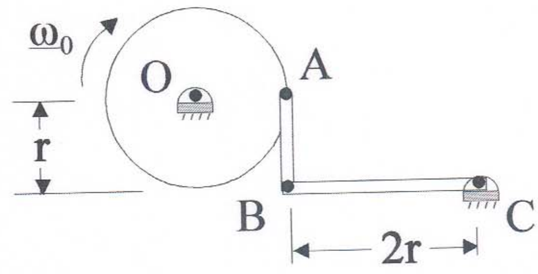


Figure 7. A kinematic mechanism.

$$\underline{\omega}_{AB} = 0 \qquad \underline{\omega}_{BC} = \frac{1}{2}\,\omega_0 \qquad \underline{\alpha}_{AB} = \frac{3}{2}\,\omega_0^{\,2} \qquad \underline{\alpha}_{BC} = 0 \tag{1}$$

a) Explain why $\underline{a}_A = r\,\omega_0^2 \leftarrow$ and $\underline{a}_C = 0$.
b) Find the linear accelerations of the centers of mass of the disk and the two rods.
c) Draw complete free body diagrams of the disk and the two rods (be sure to include M).
d) Determine the components of the forces exerted at A and B on rod AB, the reactions at O, and the applied moment, M.

a) Referring to Figure 7 and noting that point O is fixed,

$$\boxed{\underline{a}_O = 0 \ .} \tag{2}$$

The acceleration of point A can be found by considering the disk and writing the relative acceleration formula

$$\underline{a}_A = \underline{a}_O + \underline{a}_{A/O} \ . \tag{3}$$

Since the disk is spinning with a constant angular velocity, its angular acceleration is zero.

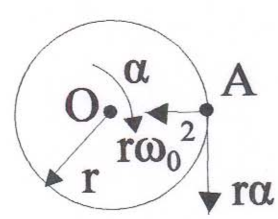


Figure 8. The disk.

Referring to Figure 8, the corresponding scalar components of Equation (3) are

$$\overset{+}{\rightarrow}\ a_{A_x} = a_{O_x} + a_{A/O_x} = 0 - r\,\omega_0^{\,2}$$

$$+\uparrow\ a_{A_y} = a_{O_y} + a_{A/O_y} = 0 - r\,\alpha = 0 - 0 = 0\ . \tag{4}$$

Hence,

$$\boxed{\underline{a}_A = r\,\omega_0^{\,2}\ \leftarrow} \tag{5}$$

b) The center of mass of the disk lies at point O. As mentioned previously, this point is fixed. Hence,

$$\boxed{\underline{a}_O = 0\ .} \tag{6}$$

The linear accelerations of the centers of mass of the rods can be determined by applying the relative acceleration formulas.

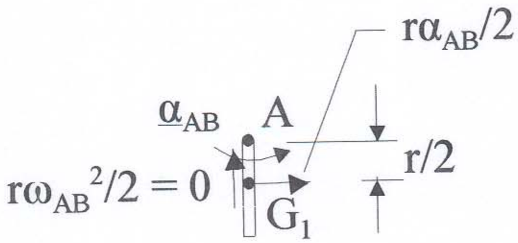


Figure 9. Rod AB.

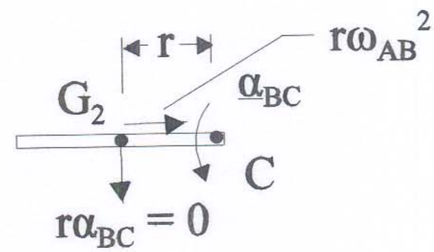


Figure 10. Rod BC.

Referring to Figure 9, for example,

$$\underline{a}_{G_1} = \underline{a}_A + \underline{a}_{G_1/O}\ . \tag{7}$$

The corresponding scalar expressions are

$$\overset{+}{\rightarrow}\ a_{G_{1_x}} = a_{A_x} + a_{G_1/A_x} = -\,r\,\omega_0^{\,2} + \frac{1}{2}\,r\,\alpha_{AB} = -\,\frac{1}{4}\,r\,\omega_0^{\,2}$$

$$+\uparrow\ a_{G_{1_y}} = a_{A_y} + a_{G_1/A_y} = 0 + 0 = 0 \tag{8}$$

and

$$\boxed{\underline{a}_{G_1} = \frac{1}{4} r \omega_0^2 \leftarrow .} \quad \textbf{(9)}$$

Referring to Figure 10,

$$\underline{a}_{G_2} = \underline{a}_C + \underline{a}_{G_2/C} . \quad \textbf{(10)}$$

Realizing that C is fixed, the corresponding scalar equations are

$$\overset{+}{\rightarrow} a_{G_{2x}} = a_{C_x} + a_{G_2/C_x} = 0 + r \omega_{BC}^2 = \frac{1}{4} r \omega_0^2$$

$$+\uparrow a_{G_{2y}} = a_{C_y} + a_{G_2/A_y} = 0 + 0 = 0 \quad \textbf{(11)}$$

and

$$\boxed{\underline{a}_{G_2} = \frac{1}{4} r \omega_0^2 \rightarrow .} \quad \textbf{(12)}$$

c) The free body diagrams for the components of the system are shown in Figure11.

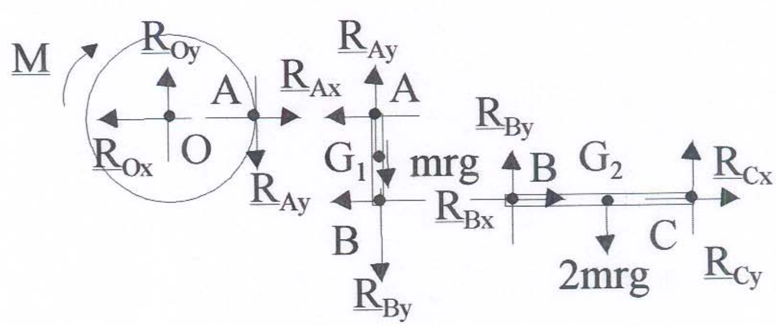


Figure 11. FBDs of all components.

Referring to the FBD of bar AB,

$$+\circlearrowleft)\ \sum M_B = \sum M_{B_{effectve}}$$

$$R_{A_x}\, r = (m_{AB}\, a_{G_1})\, \frac{r}{2} + I_{G_1}\, \alpha_{AB} = m\, r\, (\frac{1}{4}\, r\, \omega_0^{\,2})\, (\frac{r}{2}) + \frac{1}{12}\, (m\, r)\, r^2\, (\frac{3}{2}\, \omega_0^{\,2}) \qquad \textbf{(13)}$$

and

$$\boxed{\underline{R}_{A_x} = \frac{1}{4}\, m\, r^2\, \omega_0^{\,2} \leftarrow\ .} \qquad \textbf{(14)}$$

$$\overset{+}{\rightarrow}\ \sum F_x = m\, a_{G_{1_x}}$$

$$-\, R_{A_x} - R_{B_x} = m_{AB}\, a_{G_1} \qquad \textbf{(15)}$$

$$-\, \frac{1}{4}\, m\, r^2\, \omega_0^{\,2} - R_{B_x} = m\, r\, (-\, \frac{1}{4}\, r\, \omega_0^{\,2})$$

and

$$\boxed{\underline{R}_{B_x} = 0\ .} \qquad \textbf{(16)}$$

Referring to the FBD of rod BC,

$$+\circlearrowleft)\ \sum M_C = \sum M_{C_{effectve}}$$

$$r\, m_{BC}\, g - R_{B_y}\, (2\, r) = (m_{BC}\, a_{G_2})\, (0) + I_{G_2}\, \alpha_{BC} = 0 \qquad \textbf{(17)}$$

$$R_{B_y} = m\, r\, g\ .$$

Hence,

$$\boxed{\underline{R}_{B_y} = m\, r\, g \downarrow \qquad on\ rod\ AB\ .} \qquad \textbf{(18)}$$

On rod AB

$$+\uparrow \quad \sum F_y = m\, a_{G_{1_y}}$$

$$R_{A_y} - m_{AB}\, g - R_{B_y} = 0 \tag{19}$$

and

$$\boxed{\underline{R}_{A_y} = 2\, m\, g\, r \ \uparrow \ .} \tag{20}$$

On the disk

$$+\circlearrowleft \quad \sum M_O = I_O\, \alpha = 0$$

$$M + r\, R_{A_y} = 0 \tag{21}$$

and

$$\boxed{\underline{M} = 2\, m\, g\, r^2 \ \circlearrowleft \ .} \tag{22}$$

$$\overset{+}{\rightarrow} \quad \sum F_x = m\, a_{O_x} = 0$$

$$R_{A_x} - R_{O_x} = 0 \tag{23}$$

and

$$\boxed{\underline{R}_{O_x} = \frac{1}{4}\, m\, r^2\, \omega_0^{\,2} \ \leftarrow \ .} \tag{24}$$

$$+\uparrow \quad \sum F_y = m\, a_{O_y} = 0$$

$$R_{O_y} - R_{A_y} = 0 \tag{25}$$

and

$$\boxed{\underline{R}_{O_y} = 2\, m\, g\, r \;\uparrow\;.} \tag{26}$$

Note: The moment found in Equation (22) is equal but opposite to the moment applied to the disk.

Chapter 17. Plane Motion of Rigid Bodies - Energy and Momentum Methods

17.1 Basic Principle of Work and Energy

The principle of work and energy may be used to solve problems involving displacements and velocities. In general,

$$U_{1-2} = T_2 - T_1 \qquad or \qquad T_1 + U_{1-2} = T_2 \tag{17.1-1}$$

where T is the kinetic energy of the particles forming the rigid body and U is the work of all forces acting on the various particles of the body.

In a problem containing several rigid bodies, the principle of work and energy can be applied to each body, or can be written for the system as a whole. In problems involving pin-connected members, blocks and pulleys connected by inextensible cords, or meshed gears, the work may be calculated by considering only the external forces.

17.2 Work of Forces Acting on a Rigid Body

Recall,

$$U_{1-2} = \int_{s_1}^{s_2} F \cos\alpha \, ds \ . \tag{17.2-1}$$

The work done by a couple is given by,

$$U_{1-2} = \int_{\theta_1}^{\theta_2} M \, d\theta \tag{17.2-2}$$

and if M is constant,

$$U_{1\rightarrow 2} = M\,(\theta_2 - \theta_1) \quad . \tag{17.2-3}$$

The principle of work and energy can be applied in the general case of rigid body motion, for example, when a body is rolling without sliding. In this case, the friction force at the point of contact does no work.

17.3 Kinetic Energy of a Rigid Body in Plane Motion

Recall that general plane motion can be considered as the superposition of a translation followed by a rotation. In the case of a translation, all of the points in the rigid body move with the same velocity and

$$T = \frac{1}{2}\, m\, v_G^{\;2} \tag{17.3-1}$$

where v_G is the velocity of the center of mass. For a rotation about an axis passing through the point 0,

$$T = \frac{1}{2}\, I_O\, \omega^2 \tag{17.3-2}$$

where I_O is the moment of inertia about point 0.

In general,

$$\boxed{T = \frac{1}{2}\, I_C\, \omega^2 = \frac{1}{2}\, m\, v_G^{\;2} + \frac{1}{2}\, I_G\, \omega^2} \tag{17.3-3}$$

where C is the instantaneous center, and v_G and I_G are measured with respect to the mass center.

17.4 Conservation of Energy

When a rigid body moves under the action of conservative forces, the sum of the kinetic energy and the potential energy of the system remains constant and,

$$T_1 + V_1 = T_2 + V_2 \quad . \tag{17.4-1}$$

17.5 General Principle of Impulse and Momentum

The principle of impulse and momentum may be applied to solve problems involving velocities and time. Moreover, the principle of impulse and momentum provides the only practicable method for the solution of problems involving impulsive motion or impact. In general,

$$Syst\ Momenta_1 + Syst\ Ext\ Imp_{1\text{-}2} = Syst\ Momenta_2 \quad . \tag{17.5-1}$$

Equation (17.5-1) is a vector equation; the momenta is equivalent to the vector sum of the linear momentum vector $m\underline{v}_G$, attached at the center of mass, and the angular momentum couple, $I_G\underline{\omega}$.

The angular momentum about an arbitrary point, O, is

$$\underline{H}_O = I_G\, \underline{\omega} + \sum \underline{r}_{OG} \times m\, \underline{v}_G \quad . \tag{17.5-2}$$

APPENDIX I - REVIEW OF STATICS

A.1 Vectors

Composition refers to the addition of vectors and is performed using the parallelogram law or the triangle rule. Composition of three or more vectors requires successive application of the parallelogram law or the polygon rule.

Resolution, on the other hand, refers to the separation of a vector into its components.

The laws of sines and cosines are often used when composing and/or resolving forces. Referring to Figure 1:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

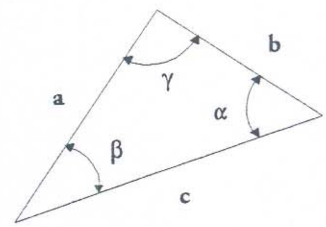


Figure 1.

and

$$c^2 = a^2 + b^2 - 2\,a\,b \cos \gamma \ .$$

Similar relationships can be written for sides a and b.

A.2 2-D Rectangular Coordinates

One of the most effective methods for working with concurrent coplanar forces is to resolve the forces into components along orthogonal coordinate axes. Referring to Figure 2:

$$F_x = F \cos \theta_x \qquad F_y = F \sin \theta_x$$

$$F = \sqrt{F_x^2 + F_y^2} \qquad \tan \theta_x = \frac{F_y}{F_x} \ .$$

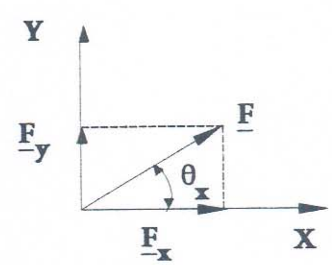


Figure 2.

When using rectangular coordinates, a scalar component is assumed to be positive when its corresponding vector is in a positive coordinate direction; otherwise, it is negative.

A.3 3-D Rectangular Coordinates

When adding three or more forces, there is no practical trigonometric solution from the polygon which describes the resultant. The best approach for finding the resultant is to resolve each force into rectangular components. The components are then composed into their resultant.

Figure 3 shows the case in which the line of action of a force is defined by two points. The force may be resolved into rectangular components using a scalar approach based on the force multiplier method. In this case:

$$\frac{F}{d} = \frac{F_x}{d_x} = \frac{F_y}{d_y} = \frac{F_z}{d_z}$$

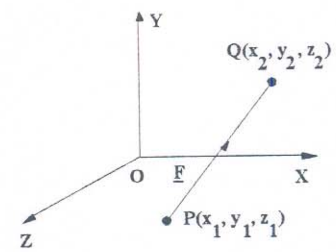


Figure 3.

where

$$d_x = x_2 - x_1 \quad d_y = y_2 - y_1 \quad d_z = z_2 - z_1 \quad d = \sqrt{d_x^2 + d_y^2 + d_z^2}$$

An alternate approach is based on unit vectors. In this case:

$$\underline{F} = F\,\lambda_{PQ} = F_x\,\underline{i} + F_y\,\underline{j} + F_z\,\underline{k}$$

where

$$\lambda_{PQ} = \frac{\overline{PQ}}{PQ} = \frac{(x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j} + (z_2 - z_1)\,\underline{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

A.4 Addition of Concurrent Forces in Space/3-D Equilibrium of Particles

When two or more forces act on the same particle, the resultant can be determined from their components; since,

$$R_x = \sum F_x \qquad R_y = \sum F_y \qquad R_z = \sum F_z$$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$\cos\theta_x = \frac{R_x}{R} \qquad \cos\theta_y = \frac{R_y}{R} \qquad \cos\theta_z = \frac{R_z}{R} \ .$$

Equilibrium exists when a particle remains at rest or moves with a constant velocity. In this case:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0 \ .$$

The method of attack is to resolve all forces into their components, sum the components and then apply the appropriate equations.

A.5 Algebraic Manipulation

The *scalar* or *dot product* of two vectors $\underline{P}$ and $\underline{Q}$ is defined by

$$\underline{P} \cdot \underline{Q} = P\,Q \cos\theta \ .$$

The result of this operation is a scalar. The dot product can be used to find the angle between two vectors, since,

$$\cos\theta = \frac{\underline{P} \cdot \underline{Q}}{P\,Q} = \underline{\lambda}_P \cdot \underline{\lambda}_Q \ .$$

It is also possible to determine the component of a vector, say $\underline{P}$, in any direction in space. The operation involves constructing a unit vector in the desired direction, say $\underline{\lambda}$, and then taking the dot product between $\underline{P}$ and $\underline{\lambda}$. Mathematically, denoting the component by P_λ,

$$P_\lambda = \underline{P} \cdot \underline{\lambda} \ .$$

In general, two vectors are perpendicular if their dot product is zero. For the unit vectors in a rectangular coordinate system:

$$\begin{array}{lll} \underline{i} \cdot \underline{i} = 1 & \underline{j} \cdot \underline{i} = 0 & \underline{k} \cdot \underline{i} = 0 \\ \underline{i} \cdot \underline{j} = 0 & \underline{j} \cdot \underline{j} = 1 & \underline{k} \cdot \underline{j} = 0 \\ \underline{i} \cdot \underline{k} = 0 & \underline{j} \cdot \underline{k} = 0 & \underline{k} \cdot \underline{k} = 1 \end{array}$$

The *vector cross product* of P̲ and Q̲, V̲, is defined as:

$$\underline{V} = \underline{P} \times \underline{Q} = P\,Q \sin\theta\, \underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \ .$$

For the unit vectors in a rectangular system:

$$\begin{array}{lll} \underline{i} \times \underline{i} = 0 & \underline{j} \times \underline{i} = -\underline{k} & \underline{k} \times \underline{i} = \underline{j} \\ \underline{i} \times \underline{j} = \underline{k} & \underline{j} \times \underline{j} = 0 & \underline{k} \times \underline{j} = -\underline{i} \\ \underline{i} \times \underline{k} = -\underline{j} & \underline{j} \times \underline{k} = \underline{i} & \underline{k} \times \underline{k} = 0 \end{array}$$

The cross product is used when computing moments; V̲ is perpendicular to the plane formed by P̲ and Q̲; V̲ has magnitude $V = P\,Q \sin\theta$ which is equal to the area of the parallelogram formed by P̲ and Q̲; and, V̲ forms a right handed triad with the vectors P̲ and Q̲.

The *scalar triple product* is defined as:

$$\underline{S} \cdot (\underline{P} \times \underline{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} .$$

The scalar triple product is equal to the volume of the parallelopiped formed by the three vectors. $\underline{S}$, $\underline{P}$ and $\underline{Q}$.

It may also be used to find the common perpendicular, say d, between $\underline{P}$ and $\underline{Q}$. This is accomplished by choosing any vector, say $\underline{r}$, which joins the lines of actions of the vectors. Then,

$$d = \frac{\underline{r} \cdot (\underline{P} \times \underline{Q})}{| \underline{P} \times \underline{Q} |} .$$

The operation can also be used to determine the moment of a force about a line AB. To obtain M_{AB}, compute a unit vector, λ_{AB} , along the line and choose a point on AB, say A. Then,

$$M_{AB} = \underline{\lambda}_{AB} \cdot \underline{M}_A = \underline{\lambda}_{AB} \cdot (\underline{r}_A \times \underline{F}) .$$

Finally, the scalar triple product can be used to determine the component of a force $\underline{F}$ perpendicular to a plane, say DAC. In this case, two noncollinear and nonparallel vectors are chosen in the plane, say **DA** and **AC**. Then,

$$F_{perpendicular} = \underline{F} \cdot \frac{(\overline{DA} \times \overline{AC})}{| \overline{DA} \times \overline{AC} |} = \underline{F} \cdot \underline{\lambda}_{perpendicular} .$$

A.6 Moments and Couples

Referring to Figure 4, the moment of a force $\underline{F}$ acting at point A about point O may be expressed as either

$$\overset{+}{\circlearrowleft}\ M = F\,d$$

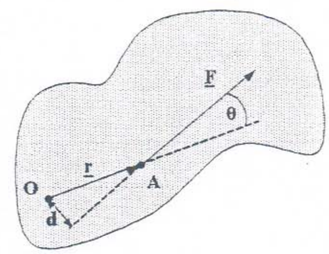


Figure 4.

or

$$\underline{M}_O = \underline{r} \times \underline{F}$$

where d is the perpendicular distance from the reference point O to the line of action of the force $\underline{F}$; and, $\underline{r}$ is a position vector drawn from O to any point along the line of action of $\underline{F}$. These equations can also be applied to compute the moment produced by the couple shown in Figure 5.

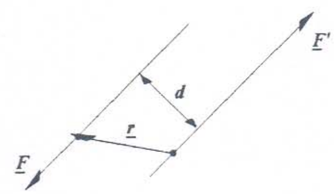


Figure 5.

Varigon's theorem states that the moment of a force about any axis is equal to the moments of its components about that axis. Conversely, the moment of several concurrent forces about any axis is equal to the sum of the moment produced by their resultant about that axis.

It should be noted that the maximum moment for a given force, or the least force necessary to produce a given moment, occurs when the force is perpendicular to the moment arm.

A.7 Force-Couple Systems

The procedure for finding the *force* in the force-couple system equivalent to the distributed forces and moment shown in Figure 6 is to move all of the forces to O and find the resultant of the resulting concurrent force system. The *moment* in the force-couple system is determined by adding the moments of all of the forces in the system about the point O to any applied couples.

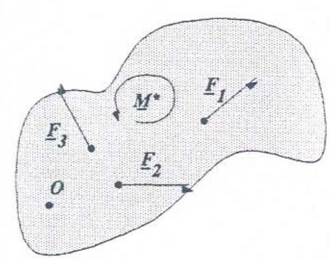


Figure 6.

When the force is perpendicular to the moment, the force-couple system can be further reduced to a single force by moving the force off its line of action until it creates a moment of the same magnitude and direction.

A.8 Equilibrium of Rigid Bodies

In general, distributed forces and applied couples cause the rigid body to translate and rotate. For the body to remain at rest or to move with a constant velocity,

$$\sum \underline{F} = 0 \qquad \sum \underline{M} = 0 \ .$$

These are vector equations with correspond to six scalar equations; namely,

$$\begin{aligned} \sum F_x &= 0 \qquad & \sum M_x &= 0 \\ \sum F_y &= 0 \qquad & \sum M_y &= 0 \\ \sum F_z &= 0 \qquad & \sum M_z &= 0 \end{aligned}$$

When dealing with a 2-D force and moment system, the equilibrium equations become:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = \sum M_A = 0$$

where A is an arbitrary point in space, usually taken in the X-Y plane. The other three equilibrium equations are automatically satisfied. Alternate equations may be applied to determine these unknowns provided that they result in independent equations. One could use, for example,

$$\sum F_x = 0 \qquad \sum M_A = 0 \qquad \sum M_B = 0$$

or

$$\sum M_A = 0 \qquad \sum M_B = 0 \qquad \sum M_C = 0 \ .$$

To eliminate unknowns and thus simplify the analytical work, select the moment center so that the line of action of one or more unknowns passes through the point and/or align an axis with one of the unknowns and take the summation of forces in the perpendicular direction.

A.9 Truss Problems

Figure 7 illustrates a *truss* which consists of straight members connected at joints. No member is continuous through a joint and all applied loads act at the joints. If the weight of the member is to be included, half of its weight is applied to each of the end joints. The force in each member is directed along the member and puts the member in either tension or compression. A tensile member pulls away from its end joints while a compression member pushes toward them. During a truss analysis, it is beneficial to assume that the force in a member always acts *away* from the end joints. If the magnitude of the force is found to be positive, the member is in tension; otherwise, it is in compression.

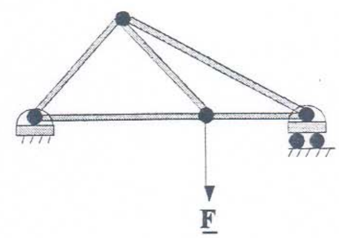


Figure 7.

The *method of joints* involves drawing a *FBD* of the entire truss and then using the 2-D equilibrium equations to solve for the external reactions at the points of constraint. The truss is initiallly inspected for *zero force members*; such members are found by inspecting the truss for unloaded joints with three connected members, two of which are collinear. In this case, the force in the third member is zero. The solution continues by starting with a joint which has a maximum of two unknown forces and applying the equilibrium equations for a concurrent system. This procedure is applied to successive joints until all unknowns are determined.

The method of joints is acceptable when the forces in all members are to be determined; however, if the forces in only a limited number of members is required, the *method of sections* may be the best approach. In some cases, especially when the truss is not *simple*, the method of sections may offer the only solution to the problem.

The method of attack is to choose a portion of the truss as a free body by passing a section through a maximum of three unknown members; apply overall equilibrium to determine the external reactions *only if necessary*; identify all zero force members; and, treat the cut members as unknown external forces and apply the conventional 2-D equilibrium equations.

An important point to remember is that if three members are cut, taking moments about the point of intersection of the lines of action of two of the members allows the force in the third member to be determined in an independent equation. Also note that the method of sections can be applied to find the force in three members and then the method of joints can be applied to find the forces in other members of interest.

A.10 Frames and Machine Problems

The *method of members* is applied to analyze structures which contain at least one member which is multiforce. That is, the member is acted upon by three or more forces, one of which is acting at a point other than a joint. The associated structures are classified either as *frames* which are usually constrained and used to support loads; or, *machines* which are to transmit or modify loads and may or may not be stationary.

The method of attack for the method of members is to first consider the structure as a whole and use external equilibrium to determine the reactions at the points of constraint; then, consider each portion of the structure as a separate free body, denoting and labeling forces consistent with Newton's third law. During this step, it is beneficial to look for two force members. When the forces and reactions are properly labeled and the various parts of the structure combined, only the external forces and the reactions remain.

A.11 Friction

Figure 8. A block rests on a horizontal surface.

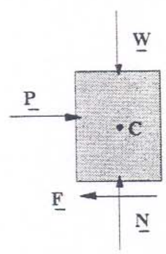


Figure 9. A progressively larger horizontal force is applied.

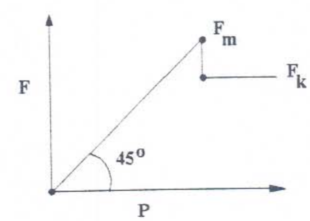


Figure 10. A plot of P versus F.

Figure 8 shows a block resting on a rough surface. The forces acting on the block are its weight W and the reaction at the surface; N is normal to the surface as a consequence of the free body diagram.

Figure 9, on the other hand, shows the case in which a small force, P, acts horizontally on the block. If $P = 0$, $F = 0$ and $W = N$ as in the previous case. If P is small enough so the block does not move, $F = P$ and F is called the static friction force. When the block is about to move $P = F_m$ (F reaches its maximum at this point). When the block moves, P drops from F_m to F_k, $F_k < F_m$, because there is less interpenetration between the irregularities of the surfaces in contact when the surfaces are moving with respect to each other. From then on, F_k, the kinetic friction force, remains constant while the velocity increases. These results are summarized in Figure 10.

From experimental tests:

$$F_m = \mu_s N \quad and \quad F_k = \mu_k N$$

where μ_s and μ_k depend on the nature of the surface, as opposed to the area; and,

$$\mu_k \simeq 0.75\ \mu_s \ .$$

Previously, P was considered horizontal, however, four different situations may occur when a force is applied to the block as shown in Figures 11 through 14.

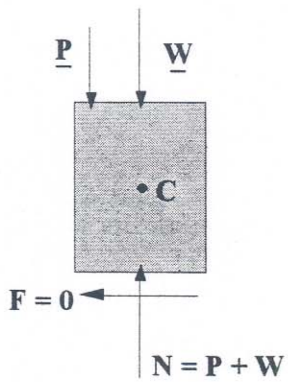


Figure 11. No friction ($P_x = 0$).

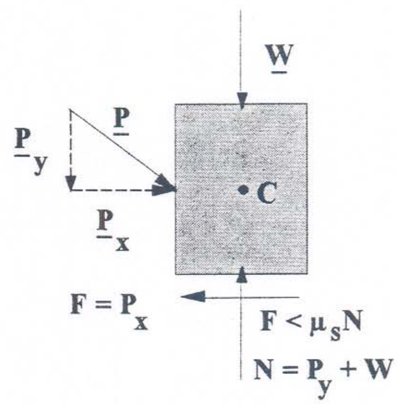


Figure 12. No motion ($P_x < 0$).

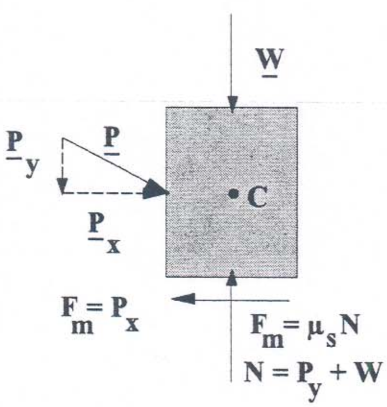


Figure 13. Motion impending ($P_x = F_m$).

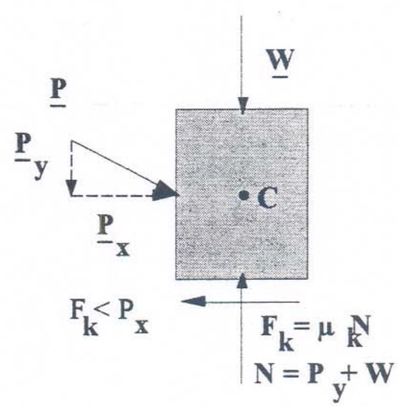


Figure 14. Motion ($P_x > F_m$).

As shown in Figures 15 through 18, it is sometimes convenient to replace F and N by their resultant R.

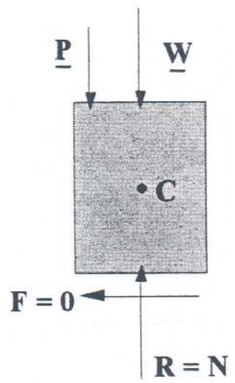


Figure 15. No friction $(P_x = 0)$.

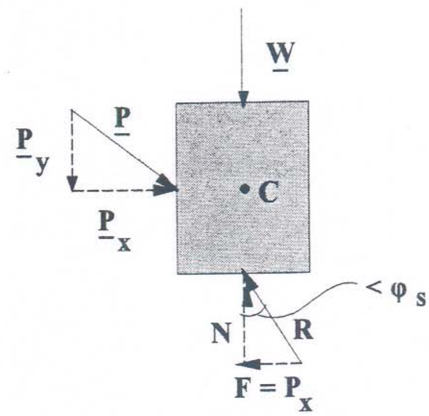


Figure 16. No motion $(P_x < 0)$.

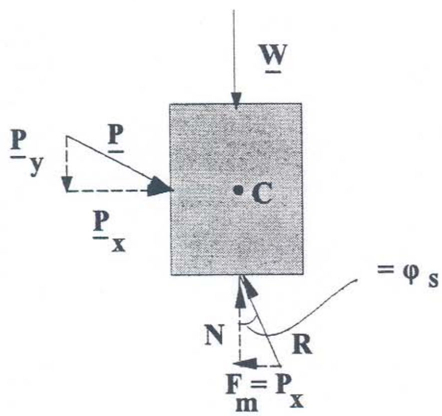


Figure 17. Motion impending $(P_x = F_m)$.

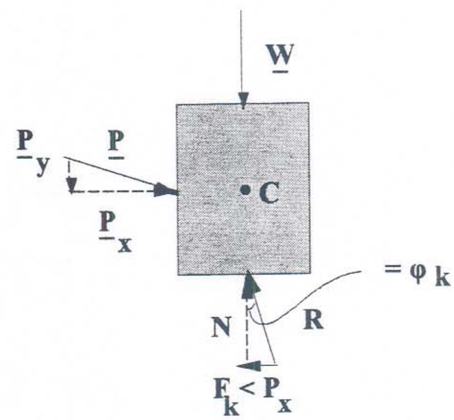


Figure 18. Motion $(P_x > F_m)$.

In Figures 16 through 18, φ_s and φ_k are the angles of static and kinetic friction, respectively; and,

$$\tan \varphi_s = \mu_s \quad \text{and} \quad \tan \varphi_k = \mu_k \ .$$

The *angle of response*, φ_s, is the angle of inclination corresponding to the condition of impending motion of a block resting on an inclined surface. The angle is obtained experimentally by placing a block on a plane which is progressively inclined. The above equation is then used to compute the coefficient of static friction.

The solution to friction problems relies on the equilibrium equations. The sense of the friction force acting on a surface is shown on the *FBD* opposite to the direction of motion. It should be noted that the minimum force required to start a block resting on a rough horizontal surface occurs when the force is perpendicular to the reaction at the surface.

To make things simple: when N can be computed directly in terms of known forces, use F and N; otherwise use R at the appropriate angle; when three forces act on a body, the law of sines, cosines, and the force triangle approach may be of benefit; if more than three forces are involved, summing forces perpendicular to the line of action of one unknown will eliminate that unknown from the equation; always consider eliminating reactions and unknown forces by taking moments about a point on their line of action.

Figure 19 illustrates the condition where a flat belt passes over a fixed cylindrical surface. The tensions in the two parts of the belt when it is about to slide to, say, the right is given by,

$$\ln \frac{T_2}{T_1} = \mu_s \beta \qquad or \qquad \frac{T_2}{T_1} = e^{\mu_s \beta}$$

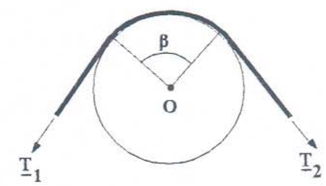


Figure 19.

where β is the angle of contact (expressed in radians) and μ_s is the coefficient of static friction. T_2 is algebraically larger than T_1; consequently, T_2 represents the tension in that part of the belt or rope which pulls, while T_1 is the tension in the portion which resists. If the belt or rope is slipping, similar relations hold true with μ_s replaced with μ_k.

A.12 Center of Gravity and Centroids

For the purposes of computing the reactions at the points of constraint, the distributed weight of a body can be considered to be a concentrated force acting through a single point called the *center of gravity*. For the flat plate shown in Figure 20:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i w_i}{W} = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i w_i}{W} = \frac{\sum_{i=1}^{n} y_i w_i}{\sum_{i=1}^{n} w_i} .$$

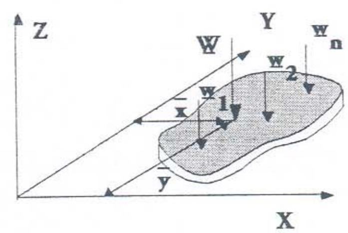


Figure 20.

For a homogeneous plate of constant thickness, the center of gravity is called the *centroid*. In this case:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{A} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i A_i}{A} = \frac{\sum_{i=1}^{n} y_i A_i}{\sum_{i=1}^{n} A_i}.$$

As the number of plates increases to infinity, the plates have differential areas and the finite sum becomes an integral. In this case,

$$\bar{x} = \frac{\int x_{el}\, dA}{A} = \frac{\int x_{el}\, dA}{\int dA}$$

$$\bar{y} = \frac{\int y_{el}\, dA}{A} = \frac{\int y_{el}\, dA}{\int dA}.$$

The expressions $\int x_{el} dA$ and $\int y_{el} dA$ are called the first moment of the area about the Y and X axis, respectively. If the centroid is located on a coordinate axis, the first moment of the area with respect to that axis is zero. The converse also holds true.

For a homogeneous wire of constant cross section:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i L_i}{L} = \frac{\sum_{i=1}^{n} x_i L_i}{\sum_{i=1}^{n} L_i}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i L_i}{L} = \frac{\sum_{i=1}^{n} y_i L_i}{\sum_{i=1}^{n} L_i}.$$

Following a similar argument to that taken for the plate:

$$\bar{x} = \frac{\int x_{el}\, dL}{L} = \frac{\int x_{el}\, dL}{\int dL}$$

$$\bar{y} = \frac{\int y_{el}\, dL}{L} = \frac{\int y_{el}\, dL}{\int dL}.$$

The above expressions specify the coordinates of the centroid of a homogeneous wire of constant cross section which coincide with its center of gravity.

Similar arguments hold true for 3-D to those derived previously in 2-D. The center of gravity of a 3-D body lies at:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i w_i}{W} = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i} \quad or \quad \frac{\int x_{el}\, dw}{W} = \frac{\int x_{el}\, dw}{\int dw}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i w_i}{W} = \frac{\sum_{i=1}^{n} y_i w_i}{\sum_{i=1}^{n} w_i} \quad or \quad \frac{\int y_{el}\, dw}{W} = \frac{\int y_{el}\, dw}{\int dw}$$

$$\bar{z} = \frac{\sum_{i=1}^{n} z_i w_i}{W} = \frac{\sum_{i=1}^{n} z_i w_i}{\sum_{i=1}^{n} w_i} \quad or \quad \frac{\int z_{el}\, dw}{W} = \frac{\int z_{el}\, dw}{\int dw} \; .$$

When the body is homogeneous:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i V_i}{V} = \frac{\sum_{i=1}^{n} x_i V_i}{\sum_{i=1}^{n} V_i} \quad or \quad \frac{\int x_{el}\, dV}{V} = \frac{\int x_{el}\, dV}{\int dV}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i V_i}{V} = \frac{\sum_{i=1}^{n} y_i V_i}{\sum_{i=1}^{n} V_i} \quad or \quad \frac{\int y_{el}\, dV}{V} = \frac{\int y_{el}\, dV}{\int dV}$$

$$\bar{z} = \frac{\sum_{i=1}^{n} z_i V_i}{V} = \frac{\sum_{i=1}^{n} z_i V_i}{\sum_{i=1}^{n} V_i} \quad or \quad \frac{\int z_{el}\, dV}{V} = \frac{\int z_{el}\, dV}{\int dV} \; .$$

The center of gravity or centroid is determined using either the *method of integration* or the *method of composites*. When the method of composites is used, a rectangular or polar coordinate axes system is established, The coordinates of the centroid of each individual portion of the area are assigned as positive or negative depending upon their position with respect to the coordinate axes system. Areas and/or lengths are considered as positive or negative depending upon whether the shape must be added or subtracted, respectively, to form the overall shape desired.

In general, it is necessary to perform a double or a triple integration to evaluate the integrals in question. In two dimensions, the differential area in rectangular coordinates is (dxdy); whereas, in polar coordinates, it is (rdrdθ). In many cases, however, a single integration can be used. In the case of an area, for example, the method of attack is to choose a rectangular strip, or appropriate sector, as an element. For a strip, the coordinates of the centroid (x_{el},y_{el}) and the area (dA) are expressed in terms of the coordinates and their differentials. In three dimensions, a similar approach can be applied if the body is one of revolution. In this case, a thin disk is taken as a differential element of volume.

A.13 Moments of Inertia

Inertia has significance when used in conjunction with other quantities (equations of motion in dynamics, flexure formula for beam stress in mechanics of materials, etc.). Basically, the term inertia is used to describe the tendency of matter to show resistance to change.

Referring to the X and Y axes shown in Figure 2, the area moments of inertia are:

$$I_x = \int y^2 \, dA \qquad I_y = \int x^2 \, dA$$

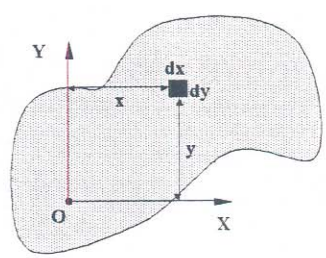


Figure 21.

The units associated with the area moment of inertia are in^4, cm^4, etc. Since $\rho^2 > 0$, the moment of inertia is always positive.

The polar moment of inertia is used to characterize torsional behavior in cylindrical shafts, the rotation of slabs, etc. It is computed for an axis passing through point O perpendicular to the x,y plane (i.e., around a z axis). By definition,

$$J_O = \int r^2 \, dA = I_x + I_y$$

where r is the distance from the element of area to the point O.

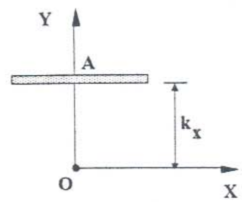


Figure 22.

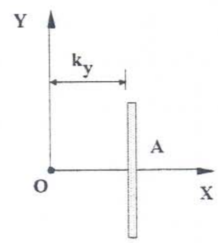


Figure 23.

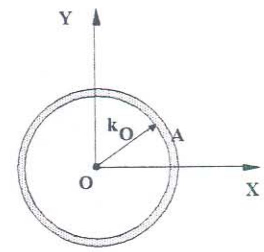


Figure 24.

Imagine that the area shown in Figure 21 is squeezed either into a strip oriented parallel to one of the coordinate axes as shown in Figures 22 and 23, or, into a thin ring centered at O as shown in Figure 24. For the area to have the same I_x, I_y and J_o, the respective strips must lie at distances k_x, k_y and k_o respectively such that,

$$I_x = k_x^2 A \qquad I_y = k_y^2 A \qquad J_O = k_o^2 A \ .$$

The distances k_x, k_y and k_o are called the radii of gyration. Note that k_x and k_y are measured perpendicular to the x and y axes, respectively, not along them.

As illustrated by Figure 25, it is often convenient to transfer the moment of inertia between a centroidal axis and an arbitrary parallel axis. This transformation can be accomplished using a simple formula known as the parallel axis theorem:

$$I = \bar{I} + A\, d^2$$

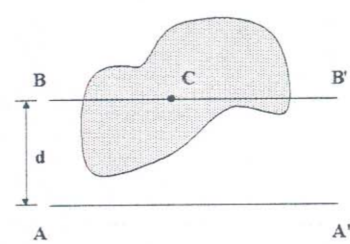


Figure 25.

where I is the area moment of inertia about the arbitrary axis, $\bar{I}$ is the moment of inertia about a parallel axis passing through the centroid of the area, and d is the perpendicular distance between the two parallel axes. Utilizing the expressions for the radii of gyration, the parallel axis theorem may also be expressed as follows:

$$k^2 = \bar{k}^2 + d^2 \ .$$

This approach can also be taken to transfer the polar moment of inertia and the polar radius of gyration, respectively, as follows:

$$J_O = \bar{J}_O + A\,d^2 \qquad\qquad k_O^2 = \bar{k}_O^2 + d^2 \ .$$

The most simple method for direct integration of the moment of inertia is to choose a strip parallel to the axis under consideration (for I_x choose the strip parallel to the x axis; for I_y choose the strip parallel to the y axis), so that the perpendicular distance to each point on the element is the same. Then express the area of the strip, dA, and either x_{el} or y_{el} as needed in terms of the coordinates of the problem and integrate.

Note: The moment of inertia can be found about one axis by integration, after which it can be transferred to a parallel axis using the parallel axis theorem. It is necessary, however, that one of the axes be centroidal.

If the area under consideration is made up of several parts, its moment of inertia about an axis is found by adding the moments of inertia of the composite areas about that axis. In general, it is necessary to transfer each moment of inertia to the appropriate axis by the parallel axis theorem *before* adding.

Note: The radius of gyration is *not* the sum of the component areas' radius of gyration. It is first necessary to perform the superposition of the moments of inertia and then use the definition the $k^2 = I_{total}/A_{total}$.

The mass moment of inertia is given by,

$$\boxed{I = \int r^2\,dm \qquad I = k^2\,m \ .}$$

In this case, the parallel axis theorem takes the form,

$$\boxed{I = \bar{I} + m\,d^2 \ .}$$

A.14 Shear and Moment Diagrams

A *beam* is a structural member which is designed to support loads at various points along its span. In most cases, loads are perpendicular to the beam. A portion of mechanics of materials deals with selecting a cross section to resist these shearing forces and bending moments. This often requires drawing shear and moment diagrams for the beam.

The method of attack is to determine the reactions at points of constraint, then cut the beam at various sections along the span, indicating on which portion of the beam internal forces are acting. A convention must be established; the shear V and the bending moment M at a given point of a beam are assumed to be positive when the internal forces and couples acting on each element of the beam are directed as shown in Figure 26.

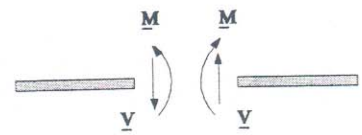


Figure 26.

The following relationships are often helpful:

$$\frac{dV}{dx} = -w$$

$$\frac{dM}{dx} = V$$

where w is the distributed load per unit length.

APPENDIX II - SOLVING ENGINEERING PROBLEMS

A.1 Introduction

Many engineering students feel that statics and dynamics are the most difficult courses that they have had. In general, it is not because the material is so difficult to understand but because these courses require a systematic approach to problem solving. The objective of the section is to help develop a comprehensive and flexible approach which may be applied to solve every new engineering problem.

A.2 McMaster Strategy

Several years ago the McMaster University engineering faculty were concerned that their students were not adequately prepared to deal with problems that they would eventually face in industry. In an effort to remedy this, the faculty conducted a study into problem solving techniques. One of the things that came out of their study was the following six step plan:

1. Always think that you want to, and that you can, solve the problem at hand. Motivate yourself and minimize distress. Your goals are to relax and build self confidence.

2. Define the problem as stated. Understand the words, identify the objectives and draw diagrams where applicable. Identify the system along with stated input, output, knowns, unknowns, stated and inferred constraints and criteria. Your goal is to fully understand the problem before attempting a solution.

3. Define and explore the issues. Draw from past experiences. Recall theory and fundamentals that seem pertinent. Hypothesize, visualize, idealize, generalize and simplify. Your goal is to collect all of your resources to attack the problem at hand.

4. Plan the course of action. Select tactics, assemble resources, develop tasks, sub-tasks, etc. Your goal is to organize the work into manageable tasks and establish an overall game plan.

5. Solve the problem.

6. Review the solution. Check for reasonableness and errors. Make sure that the criteria have been satisfied. Your goal is to assess performance and identify problems in the overall approach to prevent them from recurring.

APPENDIX III - FORMULAS, EXAMS & COMPUTER SUPPLEMENT

A.1 Basic Formulas

Rectilinear Motion:

$$a = \frac{dv}{dt} \qquad v = \frac{ds}{dt} \qquad a = v\,\frac{dv}{ds} = \frac{d^2s}{dt^2}$$

$$v = v_0 + a\,t \qquad v^2 = v_0^2 + 2\,a\,s$$

$$s = v_0\,t + \frac{1}{2}\,a\,t^2$$

Projectile Motion:

$$a_x = 0 \qquad a_y = -g$$

$$v_x = v_{0_x} \qquad v_y = v_{0_y} - g\,t$$

$$x = x_0 + v_{0_x}\,t \qquad y = y_0 + v_{0_y}\,t - \frac{1}{2}\,g\,t^2$$

General Equations:

$$a_n = r\,\omega^2 = \frac{v^2}{r} \qquad a_t = \frac{dv}{dt} = r\,\alpha \qquad v = r\,\omega$$

$\underline{v}_B = \underline{v}_A + \underline{v}_{B/A}$ *A is the reference point* $\underline{a}_B = \underline{a}_A + \underline{a}_{B/A}$ *A is the reference point*

$\sum \underline{F} = m\,\underline{a}_G$ *G is the center of mass* $\sum \underline{M}_G = I_G\,\underline{\alpha}$ *G is the center of mass*

$\sum \underline{M}_O = I_O\,\underline{\alpha}$ *or* $= \sum \underline{M}_{O_{eff}} = I_G\,\underline{\alpha}$ + *moment of the quantity* $m\,\underline{a}_G$ *about O*

Work-Energy:

$$\textit{Work of a force} = \underline{F} \cdot \underline{d} \qquad \textit{Work by a couple} = \underline{M} \cdot \underline{\theta}$$

$$\textit{Energy} = \frac{1}{2}\, m\, v^2 \ \ (\textit{translation}) \ = \frac{1}{2}\, I\, \omega^2 \ \ (\textit{rotation}) \ = m\, g\, h \ \ (\textit{potential}) \ = \frac{1}{2}\, k\, x^2 \ \ (\textit{spring})$$

Impulse-Momentum:

$$m_A\, v_A + m_B\, v_B = m_A\, v_A' + m_B\, v_B'$$

$$e = \frac{(v_B' - v_A')}{(v_A - v_B)}$$

$$e = 1 \ \ (\textit{elastic}) \qquad \textit{and} \qquad e = 0 \ \ (\textit{plastic})$$

Information Sheet for Exam No. 1

MAE/CE 362 Dynamics Exam No. 1 Beer and Johnston Chapter 11

General guidelines for Exam No. 1 are:

- The exam will last approximately 80 minutes.
- The exam will be closed book and no formula sheets will be provided.
- The exam will consist of 20 problems each worth 5 points.
- Problems will be graded based on the answer specified in the box provided. Partial credit may be given at the discretion of the instructor.
- To receive full credit, the appropriate units must be included in the final answers when applicable.
- No conversions between U.S. and MKS units will be required unless specifically asked to do so in the problem statement.

The *concepts* to be tested include:

Fundamental principles, definitions, and units.

Position, velocity, and acceleration of a particle in rectilinear motion.

Uniform and uniformly accelerated rectilinear motion of a particle.

Relative motion of several particles moving in rectilinear motion.

Position, velocity, and acceleration of a particle in curvilinear motion.

Projectile motion.

Relative motion of several particles moving in curvilinear motion.

Normal and tangential components of acceleration.

Radial and transverse components of acceleration.

Information Sheet for Exam No. 2

MAE/CE 362 Dynamics Exam No. 2 Beer and Johnston Chap. 12&13

General guidelines for Exam No. 2 are:

- The exam will last approximately 80 minutes.
- The exam will be closed book and no formula sheets will be provided.
- The exam will consist of 20 problems each worth 5 points.
- Problems will be graded based on the answer specified in the box provided. Partial credit may be given at the discretion of the instructor.
- To receive full credit, the appropriate units must be included in the final answers when applicable.
- No conversions between U.S. and MKS units will be required unless specifically asked to do so in the problem statement.

The *concepts* to be tested include:

Fundamental principles, definitions, and units.

Newton's 2nd law of motion.

Linear and angular momentum of a particle.

Newton's law of gravitation; motion under a conservative central force.

Work, power, efficiency, kinetic energy, and potential energy.

Applications of the principle of work and energy.

Conservative force systems; conservation of energy.

Applications of the principle of impulse and momentum.

Direct central impact; problems involving energy and momentum.

Information Sheet for Exam No. 3

MAE/CE 362 Dynamics Exam No. 3 Beer and Johnston Chap. 14&15

General guidelines for Exam No. 3 are:

- The exam will last approximately 80 minutes.
- The exam will be closed book and no formula sheets will be provided.
- The exam will consist of 20 problems each worth 5 points.
- Problems will be graded based on the answer specified in the box provided. Partial credit may be given at the discretion of the instructor.
- To receive full credit, the appropriate units must be included in the final answers when applicable.
- No conversions between U.S. and MKS units will be required unless specifically asked to do so in the problem statement.

The *concepts* to be tested include:

Application of Newton's Laws to systems of particles.

Linear and angular momentum of a system of particles.

Motion of the mass center of a system of particles.

Kinetic energy and conservation of momentum for a system of particles.

Principle of impulse and momentum for a system of particles.

Translation and rotation of a rigid body.

General plane motion of a rigid body.

Absolute and relative velocity of a rigid body in plane motion.

Instantaneous center of rotation of a rigid body in plane motion.

Absolute and relative acceleration of a rigid body in plane motion.

MAE/CE 362 Dynamics Computer Assignment Beer and Johnston

As illustrated in Figure 1, a guided missile was launched from an aircraft traveling at 900 mi/hr in level flight at an altitude of 1000 ft. The longitudinal axis of the 1000-lb missile remains horizontal and the engine produces a constant thrust, T.

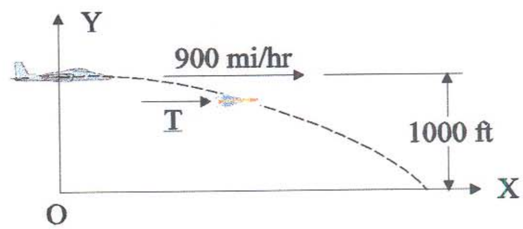


Figure 1. A missile is launched.

Review the solution below and then, using computational methods, plot the three trajectories of the missile from the time that it is released to the time that it strikes the ground for the three cases when the thrust, T, is equal to 0, 1000 lb, and 2000 lb. Include the three plots and the data sheet.

Air resistance can be neglected. Note: 5,280 ft = 1 mile; 3,600 s = 1 hr.

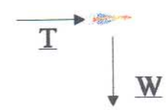


Figure 2. FBD.

Analytical Solution:

Referring to the free body diagram in Figure 2, and applying the equations of motion with W = 1000 lb,

$$\overset{+}{\rightarrow} \sum F_x = m\,a_x \quad \text{and} \quad +\uparrow \sum F_y = m\,a_y$$

$$T = \frac{1000}{32.2}\,a_x \quad \text{and} \quad -1000 = \frac{1000}{32.2}\,a_y \qquad \textbf{(1)}$$

$$a_x = \frac{32.2\,T}{1000}\,\frac{ft}{s^2} \quad \text{and} \quad a_y = -32.2\,\frac{ft}{s^2}\,.$$

Since the thrust is different but constant for each of the cases to be considered, the missile's motion is uniformly accelerated in both directions. Integrating, noting that $\underline{v}_0$ = 900 mi/hr = 1320 ft/s →,

$$v_x = v_{0_x} + a_x\,t \quad \text{and} \quad v_y = v_{0_y} + a_y\,t$$

$$v_x = 1320 + \frac{32.2\,T}{1000}\,t \quad \text{and} \quad v_y = 0 - 32.2\,t\,. \qquad \textbf{(2)}$$

Integrating once again, noting that the missile starts at $x_0 = 0$ and $y_0 = 1000$ ft,

$$x = x_0 + v_{0_x}\,t + \frac{1}{2}\,a_x\,t^2 \quad \text{and} \quad y = y_0 + v_{0_y}\,t + \frac{1}{2}\,a_y\,t^2$$

$$x = 0 + 1320\,t + \frac{1}{2}\,\frac{32.2\,T}{1000}\,t^2 \quad \text{and} \quad y = 1000 + 0 - \frac{1}{2}\,(32.2)\,t^2\,. \qquad \textbf{(3)}$$

For the three cases to be considered:

$$T = 0\ lb: \quad x = 1320\ t \quad and \quad y = 1000 - 16.1\ t^2$$

$$T = 1000\ lb: \quad x = 1320\ t + 16.1\ t^2 \quad and \quad y = 1000 - 16.1\ t^2 \tag{4}$$

$$T = 2000\ lb: \quad x = 1320\ t + 32.2\ t^2 \quad and \quad y = 1000 - 16.1\ t^2\ .$$

In all three cases, the time required for the missile to hit the ground is the same. This occurs when $y = 0$ at approximately $t = 7.88$ s. The expressions in Equation (4) may be evaluated for the range $0 < t < 7.88$ to obtain x and y; the trajectories are simply plots of y (vertical axis) versus x (horizontal axis).

Computational Solution Using Microsoft Excel:

Go to the desktop and use the icon there, or the "programs" option, and locate the icon for Excel. Click to open the spreadsheet (Block A1 on Sheet 1 should be highlighted).

Type the labels t, y, T=0 lb, T=1000 lb, and T=2000 lb into blocks A1, B1, C1, D1, and E1, respectively (Use the arrow keys to move from block to block). Refer to the attached information.

Move to block A2 and input the number 0; move down to A3 and enter 0.2; in A4 enter 0.4, etc until you reach 7.8 (Block A41). In block A42 enter 7.881104. The latter corresponds to the exact time that the missile strikes the ground.

Move to block B2. Click the "=" sign located above the spreadsheet and in the box to the right enter the text string 1000-16.1*((A2)^2). This is the expression for the altitude, y [See Equation (4)]. The box on the menu should read: =1000-16.1*((A2)^2). Now hit enter; block B3 should be highlighted and block B2 should read 1000.

Use the arrow keys to go back to block B2. Using the mouse grab the lower right hand corner and while holding the mouse button down, move to block B42, and release. The values seen on the attached sheet should result.

Move to block C2. Click the "=" sign located above the spreadsheet and in the box to the right enter the text string 1320*A2. This is the expression for the range, x, for T = 0; the box on the menu should read: =1320*A2. Hit enter; block C3 should be highlighted and block C2 should read 0.

Use the arrow keys to go back to block C2. Using the mouse grab the lower right hand corner and while holding the mouse button down, move to block C42, and release. The values seen on the attached sheet should result.

Move to block D2. Click the "=" sign located above the spreadsheet and in the box to the right enter the text string 1320*(A2)+16.1*((A2)^2). This is the expression for the range, x, for T = 1000 lb; the box on the menu should read: =1320*(A2)+16.1*((A2)^2). Hit enter; block D3 should be highlighted and block D2 should read 0.

Use the arrow keys to go back to block D2. Using the mouse grab the lower right hand corner and while holding the mouse button down, move to block D42, and release. The values seen on the attached sheet should result.

Move to block E2. Click the "=" sign located above the spreadsheet and in the box to the right enter the text string 1320*(A2)+32.2*((A2)^2). This is the expression for the range, x, for T = 2000 lb; the box on the menu should read: =1320*(A2)+32.2*((A2)^2). Hit enter; block E3 should be highlighted and block E2 should read 0.

Use the arrow keys to go back to block E2. Using the mouse grab the lower right hand corner and while holding the mouse button down, move to block E42, and release. Highlight the entire table and use the "Format Cells" option to center all entries. If printed, the table should look exactly like the one on the attached sheet. Now, for the graphs.

Highlight the block B1 through E42 using the mouse and release (not column A). The block should remain highlighted. Go up to icon on the menu entitled "Chart Wizard" and click it. Select the "XY (Scatter)" option and choose the connected-dot "sub-type." Then click "Next." Note that the X and Y data is interchanged.

To correct this problem, click "Series." The string T=0 lb should be highlighted. Change the text string in the box labeled "X values" to =Sheet1!$C$2:$C$42; and, the text string in the box labeled "Y values" to =Sheet1!$B$2:$B$42.

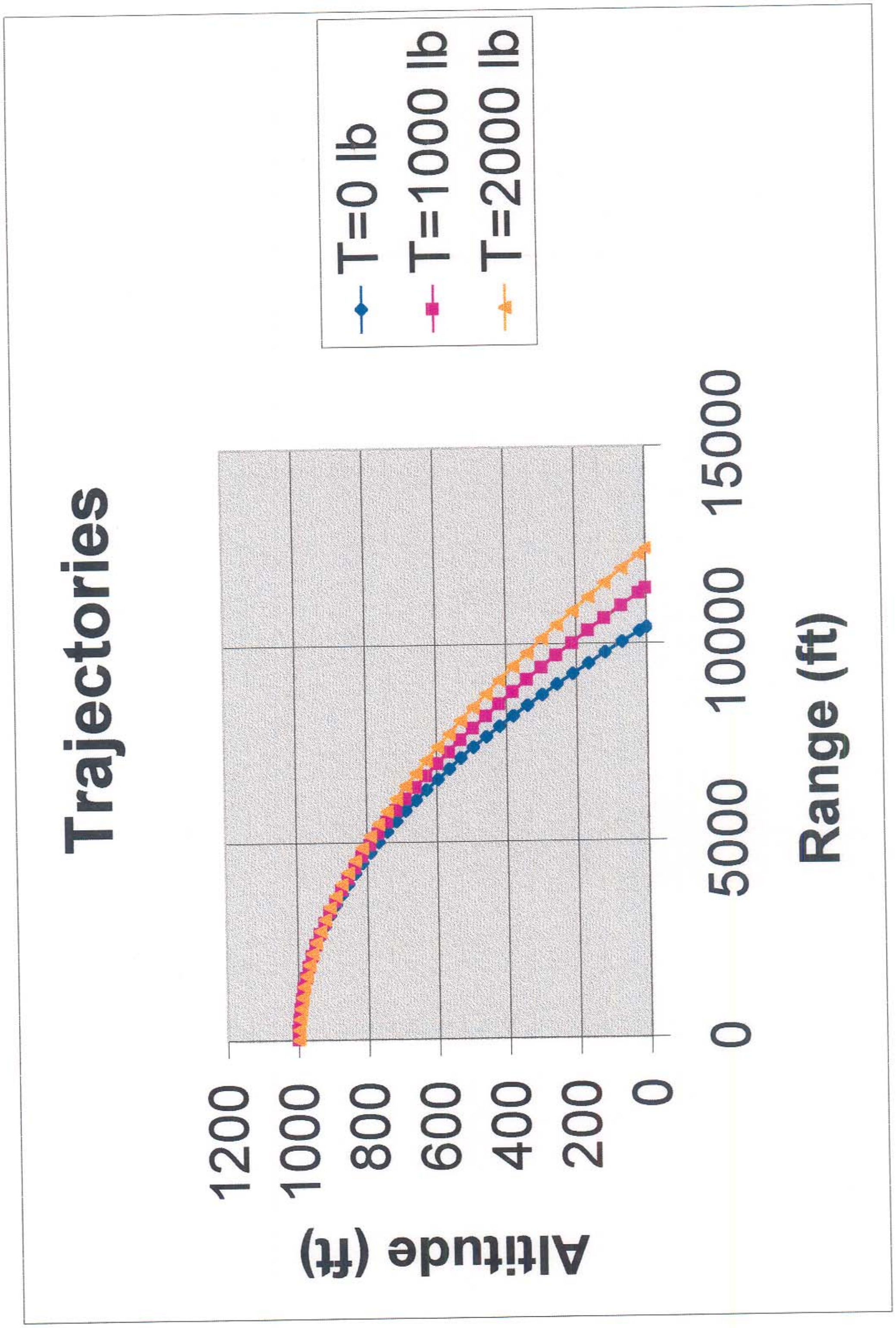
Highlight the string T=1000 lb. Change the text string in the box labeled "X values" to =Sheet1!$D$2:$D$42; and, the text string in the box labeled "Y values" to =Sheet1!$B$2:$B$42.

Highlight the string T=2000 lb. Change the text string in the box labeled "X values" to =Sheet1!$E$2:$E$42; and, the text string in the box labeled "Y values" to =Sheet1!$B$2:$B$42. All three trajectories should now appear in the proper fashion, so hit "Next."

Go to "Titles." Under "Chart Title:" type Trajectories. Under "Value (x) axis:" type Range (ft). Under "Value (y) axis:" type Altitude (ft). Go to "Gridlines." Under "Value (x) axis" click the "Major gridlines" box. Hit "Next." Now, by hitting "Finish," the graph is superimposed on the data sheet. It should already be highlighted, so simply click on "print" to complete the assignment.

EXCEL Data Sheet for Missile Problem

	A	B	C	D	E
1	t	y	T=0 lb	T=1000 lb	T=2000 lb
2	0	1000	0	0	0
3	0.2	999.356	264	264.644	265.288
4	0.4	997.424	528	530.576	533.152
5	0.6	994.204	792	797.796	803.592
6	0.8	989.696	1056	1066.304	1076.608
7	1	983.9	1320	1336.1	1352.2
8	1.2	976.816	1584	1607.184	1630.368
9	1.4	968.444	1848	1879.556	1911.112
10	1.6	958.784	2112	2153.216	2194.432
11	1.8	947.836	2376	2428.164	2480.328
12	2	935.6	2640	2704.4	2768.8
13	2.2	922.076	2904	2981.924	3059.848
14	2.4	907.264	3168	3260.736	3353.472
15	2.6	891.164	3432	3540.836	3649.672
16	2.8	873.776	3696	3822.224	3948.448
17	3	855.1	3960	4104.9	4249.8
18	3.2	835.136	4224	4388.864	4553.728
19	3.4	813.884	4488	4674.116	4860.232
20	3.6	791.344	4752	4960.656	5169.312
21	3.8	767.516	5016	5248.484	5480.968
22	4	742.4	5280	5537.6	5795.2
23	4.2	715.996	5544	5828.004	6112.008
24	4.4	688.304	5808	6119.696	6431.392
25	4.6	659.324	6072	6412.676	6753.352
26	4.8	629.056	6336	6706.944	7077.888
27	5	597.5	6600	7002.5	7405
28	5.2	564.656	6864	7299.344	7734.688
29	5.4	530.524	7128	7597.476	8066.952
30	5.6	495.104	7392	7896.896	8401.792
31	5.8	458.396	7656	8197.604	8739.208
32	6	420.4	7920	8499.6	9079.2
33	6.2	381.116	8184	8802.884	9421.768
34	6.4	340.544	8448	9107.456	9766.912
35	6.6	298.684	8712	9413.316	10114.63
36	6.8	255.536	8976	9720.464	10464.93
37	7	211.1	9240	10028.9	10817.8
38	7.2	165.376	9504	10338.62	11173.25
39	7.4	118.364	9768	10649.64	11531.27
40	7.6	70.064	10032	10961.94	11891.87
41	7.8	20.476	10296	11275.52	12255.05
42	7.881104	1.58E-05	10403.06	11403.06	12403.06


Trajectories
Altitude (ft)
1200
1000
800
600
400
200
0
0
5000
10000
15000
Range (ft)
T=0 lb
T=1000 lb
T=2000 lb