MAE/CE 370

- MECHANICS OF MATERIALS -

CLASS NOTES
Version 10.4

by

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Professor of Mechanical Engineering
University of Alabama in Huntsville

Spring 2013
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MAE/CE 370 - Mechanics of Materials
Spring 2013

2011-2012 Catalog Data: MAE/CE 370: Mechanics of Materials. 4 hrs. Topics include: theory of stress and strain, Hooke's law, analysis of stresses and deformations in bodies loaded by axial, torsional, bending, and combined loads, and analysis of statically indeterminate systems. Laboratory includes: determination of selected properties of various engineering materials, experimental verification of theories presented, use of strain measuring devices, test procedures, instrumentation, and interpretation of results. Prerequisites: MAE 285 or CPE 112 or CHE 197, and MAE/CE 271.


Laboratory: A laboratory section of MAE/CE 370 must be scheduled and passed to receive a passing grade in the course. Laboratory location: TH N275. Important messages for the laboratory instructor(s) during the lab hours can be left in the MAE Department office: (256) 824-5118; (256) 824-6154.

Section/Room: Section No. 001; MW 3:55 p.m. - 5:15 p.m.; SC 109; Instructor: J. Gilbert
Section No. 002; MW 3:55 p.m. - 5:15 p.m.; SC S121; Instructor: S. Tillman

Coordinator: John A. Gilbert, Ph.D.
Office: 301E Optics Building
Phone: (256) 824-6029 (Direct); (256) 824-5118 (Ms. Cindy Murphy)
Fax: (256) 824-6758
E-Mail: jag@eng.uah.edu

Office Hours: To be announced.

Grading: 3 In-Class Exams (2 on course material; 1 on lab) = 45%
Homework = 10%
Final Exam = 30%
Laboratory assignments, reports, and oral presentation = 15%
Objective:

The aim of Mechanical/Civil Engineering 370 (Mechanics of Materials) is to develop in the student an understanding of the behavior of materials subjected to mechanical and/or thermal loads and to prepare him/her to analyze and design simple mechanical systems. The course is designed to provide students with a firm foundation for further study in mechanics.

Approach:

Logic and discipline are the keys to success in Mechanics of Materials. Students are trained to think their way through a technical situation in a systematic manner, and that is a good exercise for anyone who wants to be a good engineer.

Requirements and Expectations:

Students are reminded that a laboratory section of MAE/CE 370 must be scheduled and passed to receive a passing grade in the course. Keep in mind that students are expected to be present from the beginning to the end of each semester, attend all classes, and take all examinations according to their assigned schedule. In case of absence, students are expected to satisfy the instructor that the absence was for good reason.

Attendance Policy and Academic Misconduct:

For excessive cutting (3 or more classes), or for dropping the course without following the official procedure, students may fail the course. Use of a solutions manual or "old problem sets" is discouraged. Unauthorized use of written materials, computers, electronic devices, and/or collaboration during an examination constitutes an act of academic misconduct.

Homework Policy:

Homework is due at the beginning of the class on the date prescribed. Homework assignments shall be done on one side only of 8 1/2" x 11" paper. Each problem shall begin on a separate page and work must be legible. Problems shall be restated prior to solution and free-body diagrams (FBDs) shall be drawn for problems requiring such. Each page shall contain the following (in the upper right hand corner): Your name, the date, and page __ of __. All final answers must be boxed and converted (SI to US or US to SI). Loose sheets shall be placed in the correct order and stapled together in the upper left hand corner.
MAE/CE 370 - Mechanics of Materials  
COURSE SCHEDULE - MW


Prerequisites: MAE 285 or CPE 112 or CHE 197, and MAE/CE 271.

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**FINAL EXAM:** Wednesday, May 1, 2013 from 3:00 p.m. - 5:30 p.m.

**Note:** The above schedule is intended as a guide. Exam dates and homework assignments are subject to change at the discretion of the instructor.
MAE/CE 370 - Mechanics of Materials
COURSE SYLLABUS

Responsible Department:
Mechanical and Aerospace Engineering

Catalog Description:
Topics include: theory of stress and strain, Hooke's law, analysis of stresses and deformations in bodies loaded by axial, torsional, bending, and combined loads, and analysis of statically indeterminate systems. Laboratory includes: determination of selected properties of various engineering materials, experimental verification of theories presented, use of strain measuring devices, test procedures, instrumentation, and interpretation of results.

Prerequisites:
MAE 285 or CPE 112 or CHE 197, and MAE/CE 271.

Textbook:

Course Objectives:
1. To develop an understanding of the behavior of materials by analyzing and designing simple mechanical systems using fundamental and well-understood principles of mathematics, science, and engineering.
2. To provide students with a firm foundation for further study in mechanics.

Topics Covered:
1. Introduction to mechanics of materials
2. Concept of stress
3. Stress and strain under axial loading
4. Torsion
5. Pure bending
6. Analysis and design of beams for bending
7. Shear stresses in beams
8. Transformations of stress and strain
9. Deflection of beams
10. Columns
11. Energy methods

Class Schedule:
Twice weekly for sixteen weeks; classes 80 minutes each.

Contribution of Course to Meeting the Professional Component:
Basic Mathematics & Science: 0 credits.
Engineering Science: 4 credits.
Engineering Design: 0 credits.
Relationship of Course to Program Outcomes:

1) To develop within our students an ability to:
   
a) apply knowledge of mathematics, science, and engineering;

b) apply a knowledge of calculus-based physics;

c) to apply advanced mathematics through multivariate calculus and
differential equations;

d) to apply a knowledge of statistics and linear algebra; exam number 3
   (See pages 486-490, 594-601, 662-663, and laboratory manual.)

f) work professionally in thermal and mechanical systems;

g) use the techniques, skills, and modern engineering tools necessary for
   engineering practice;

h) conduct experiments and to analyze and interpret data.

Person Preparing this Description:
John A. Gilbert, Ph.D., Course Coordinator, Professor
MAE Program
January 1, 2013
CHAPTER 1 - INTRODUCTION TO STRESS

1.1 Introduction

The study of statics and dynamics was reduced to the analysis of simple mathematical models (particles and rigid bodies) which obeyed a few fundamental laws (Newton’s laws). The study of mechanics of materials, however, often relies on a variety of formulas, the validity of which depends upon assumptions seldom understood or remembered by students.

The main objective of the study of mechanics of materials is to provide an engineer with the means of analyzing and designing various machines and load bearing structures. Stress is a measure of force intensity and one of the quantities that help to define structural integrity.

1.2 Axial Loading and Normal Stress

Consider the frame shown in Figure 1. We would like to know whether or not the given load can be supported. This depends on the force intensity (stress) in the members, as well as an investigation of the stress in the pins and their bearings.

![Figure 1](image1.png)  
**Figure 1.** A frame is subjected to a load of 30 kN applied at point B.

![Figure 2](image2.png)  
**Figure 2.** The free body diagram (FBD) corresponding to Fig. 1.

The approach developed in statics is applied to obtain the free body diagram (FBD) shown in Figure 2; note that AB and BC are 2-force members. From geometry,

\[
\tan \theta = \frac{1.5}{2} \quad \theta = 36.87^\circ.
\]  

(1)
The equilibrium equations are now applied as follows:

\[ + \uparrow \sum F_y = 0 = F_{BC} \sin \theta - 30 \]
\[ F_{BC} = 50 \text{kN} \]

\[ + \rightarrow \sum F_x = 0 = -F_{BC} \cos \theta - F_{AB} \]
\[ F_{AB} = -F_{BC} \cos \theta = -40 \text{kN} \]  

The determination of the forces in the members of the frame is a first step in establishing structural integrity. Assume, for example, that member BC (50 kN tension) has a circular cross section as depicted in Figure 3.

**Figure 3.** The force (left) and stress (right) acting on bar BC of the frame shown in Fig. 1.

The stress, or force intensity, is given by,

\[ \sigma = \frac{P}{A} \]  

(1.2-1)

where \( P \) is the force, labeled as \( F_{BC} \) in Figure 3, and \( A \) is the cross sectional area.

The rod is under axial loading and the stress is perpendicular to the plane passed through the cross section. A stress directed normal to an exposed area is called a normal stress. By definition, a positive normal stress (tensile) acts away from the exposed surface while a negative normal stress (compressive) acts towards the exposed surface.

In SI units, stress is expressed in N/m\(^2\) which is defined as a pascal (Pa). Since most stresses are large, kilopascals, megapascals or gigapascals are often used, where:
In U.S. units, stress is expressed in lb/in$^2$ which is usually written as psi. A conversion between units can be made using,

\[ 1 \text{ psi} = 6.895 \text{ kPa} \quad \text{or} \quad 1 \text{ ksi} = 6.895 \text{ MPa} \]. \hspace{1cm} (1.2-3)

Example:

Assume that in Figure 1, rod BC is made of a steel that has a maximum allowable stress of 165 MPa = 23.9 ksi. If the rod has a diameter of 20 mm,

\[ P = F_{bc} = 50 \text{ kN} = 50 \times 10^3 \text{ N} \]

\[ A = \pi r^2 = \pi \left(\frac{20 \text{ mm}}{2}\right)^2 = \pi \left(10 \times 10^{-3} \text{ m}\right)^2 = 314 \times 10^{-6} \text{ m}^2 \]

\[ \sigma = \frac{P}{A} = \frac{50 \times 10^3 \text{ N}}{314 \times 10^{-6} \text{ m}^2} = 159 \times 10^6 \text{ Pa} = 159 \text{ MPa} \]

Since the stress in the rod (159 MPa) is less than the maximum allowable stress that the material can withstand (165 MPa), the rod will adequately support the load.

Equation (1.2-1) can also be used for design purposes. It is possible, for example, to determine the minimum diameter of BC required to support the applied load.

Example:

To illustrate how this is done for a different material, let’s make the new assumption that the bar is made of aluminum, as opposed to steel. If the allowable stress in the aluminum is 100 MPa,

\[ \sigma_{all} = \frac{P}{A} \quad A = \frac{P}{\sigma_{all}} = \frac{50 \times 10^3 \text{ N}}{100 \times 10^6 \text{ Pa}} = 500 \times 10^{-6} \text{ m}^2 . \]

Since the cross section is circular having an area, \( A = \pi r^2 \),

\[ r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{500 \times 10^{-6} \text{ m}^2}{\pi}} = 12.62 \times 10^{-3} \text{ m} = 12.62 \text{ mm} \] \hspace{1cm} (6)

and

1.3
\[ d = 2 \, r = 25.2 \, mm = 0.99 \, in. \] (7)

The stress computed above is the average stress over the cross section. In general, the stress at any point is given by

\[ \sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}. \] (1.2-4)

In general, stress varies across the section. Consider, for example, the bar shown in Figure 4.

\[ \text{Figure 4. Stress tends to flow through the structure and equalizes over cross sections removed from the applied loads.} \]

Stress tends to flow through the member and in this case, at a section away from the cross section, becomes fairly uniform. In all cases,

\[ P = \int dF = \int_A \sigma \, dA. \] (1.2-5)

The actual distribution of stresses in any given section is statically indeterminate and it becomes necessary to consider the deformations resulting from the loading. In the real world, stresses can build up to cause a stress concentration which can significantly weaken a portion of the structure. In complicated situations, these stresses must be computed using finite element methods, or measured using experimental techniques.

In mechanics of materials, it will be assumed that the distribution of normal stresses in an axially loaded member is uniform, except in the vicinity of an applied load (valid for all straight 2-force members in trusses and pin-connected frames). In this case, the resultant passes through the centroid of the section.
It should be noted that a uniform distribution of stress is possible only if the line of action of the concentrated loads passes through the centroid of the section considered. When this is not the case (eccentric loading as discussed in Chapter 4), bending stresses result and the stress is no longer uniform.

Example:

In Figure 5, the central portion of rod BE has a uniform rectangular cross section of 12 x 25 mm. Determine the magnitude of the forces for which the normal stress in that portion of BE is 90 MPa. The dimensions are given in meters.

Since member BE is a straight, two-force member, the force can be computed using Eqn. (1.2-1) as follows:

\[
F_{BE} = A_{BE} \sigma_{BE} = [0.012 \times 0.025] \times (90 \times 10^6) = 27 \times 10^3 \text{ N} = 27 \text{ kN} . \tag{1}
\]

A FBD is shown as Figure 6. Note that Newton's third law is satisfied at point C; the load that is applied at this point may be located on either member ABC or member CD. Applying equilibrium to the FBD labeled as 1 in Figure 6:

\[
+ \sum M_A = 0 = 0.15 F_{BE} - 0.35 P - 0.45 P - 0.45 R_{Cy} \tag{2}
\]

and

\[
0.15 F_{BE} - 0.8 P - 0.45 R_{Cy} = 0 . \tag{3}
\]

From the FBD labeled as number 2:

\[
+ \sum M_D = 0 = 0.15 P - 0.25 R_{Cy} \quad R_{Cy} = 0.6 P . \tag{4}
\]
1.6

Substituting Eqn. (4) into Eqn. (3):

\[ 0.15 \times (27) - 0.8 P - 0.45 \times (0.6 P) = 0 \quad P = 3.79 kN = 0.85 kips. \quad (5) \]

1.3 Shearing Stress

Shearing stresses are caused by the application of transverse loads and are commonly found in bolts, pins and rivets used to connect various structural members. For example, the upper portion of Figure 7 shows two plates A and B and a rivet CD.

![Figure 7](image)

**Figure 7.** The two plates cause the rivet to be in single shear along the interface EE'.

As illustrated in the lower left portion of the figure, the plates create two diametrically opposed forces that act on the rivet at the interface EE'. Since there is only one shear plane, the rivet is in single shear. As illustrated in the lower right portion of the figure; on section EE', \( P = F \). Hence, the average shear stress is:

\[ \tau_{ave} = \frac{P}{A} = \frac{F}{A} \quad . \quad (1.3.1) \]

A rivet can also be in double shear. This is illustrated in the upper portion of Figure 8.

![Figure 8](image)

**Figure 8.** The rivets are both in double shear, since shear forces occur along KK' and LL'.
The lower left hand portion of the figure illustrates that there are two shear planes in rivet HJ. The average shear stress on each of the interfaces in the rivet is calculated from the free body diagram of the section lying between planes KK’ and LL’ (shown in the lower right hand portion of the figure). In this case, \( P = F/2 \), and

\[
\tau_{ave} = \frac{P}{A} = \frac{F}{2A}.
\]  

(1.3.2)

1.4 Bearing Stress

Bearing stresses are created between surfaces in contact and are obtained by dividing the applied load by the projected contact area. Figure 9, for example, shows rivet CD of Figure 7.

![Figure 9](image.png)

**Figure 9.** The bearing stress is computed based on the projected area.

The bolt exerts on plate A a force, \( P \), equal and opposite to the force, \( F \), exerted by the plate on the bolt. In this case, the projected area is equal to \( td \), where \( d \) is the diameter of the rivet, and

\[
\sigma_b = \frac{P}{A} = \frac{P}{td}.
\]  

(1.4.1)

In general, one must first determine the forces in structural members and then compute axial, shear, and bearing stresses in members, pins, etc.

**Example:** See Sample Problem 1.1 on page 18 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.

1.7
1.5 Stress on an Oblique Plane under Axial Loading

Axial and transverse forces cause both normal and shearing stresses on planes that are not perpendicular to the axis of a member. Figure 10(a), for example, shows a section that passes through a member at an angle \( \theta \); the force, \( P \), acts on the oblique plane as shown in Figure 10(b).

Figure 10. The normal and shear stress distribution at a point depend upon the orientation of the plane under consideration.

Figure 10(c), on the other hand, illustrates that this force can be resolved into two components \( F \) and \( V \), normal and tangent to the plane, respectively. These components are given by:

\[
F = P \cos \theta \\
V = P \sin \theta
\]  

(1.5.1)

The normal and shear stresses on the inclined plane are given by:

\[
\sigma = \frac{F}{A_0} \\
\tau = \frac{V}{A_0}
\]  

(1.5.2)

where

\[
A_0 = A_\theta \cos \theta
\]  

(1.5.3)

Solving for \( A_0 \) from Eqn. (1.5-3) and substituting into Eqn. (1.5-2) yields,
\[
\sigma = \frac{P \cos \theta}{A_0 / \cos \theta} \quad \tau = \frac{P \sin \theta}{A_0 / \cos \theta} \quad \text{(1.5.4)}
\]

or,
\[
\sigma = \frac{P}{A_0} \cos^2 \theta \quad \tau = \frac{P}{A_0} \sin \theta \cos \theta \quad \text{(1.5.5)}
\]

By differentiating the expression for \(\sigma\) in Eqn. (1.5-5) with respect to \(\theta\) and setting the result equal to zero, it is determined that the maximum value of the normal stress, \(\sigma_m\), occurs on the plane oriented at \(\theta = 0^\circ\). The stresses that act on this plane [shown in Figure 11(b)] are obtained by substituting \(\theta = 0^\circ\) into the expressions in Eqn. (1.5-5).

\[
\sigma_m = \frac{P}{A_0} \quad \tau = 0 \quad \text{(1.5.6)}
\]

By differentiating the expression for \(\tau\) in Eqn. (1.5-5) with respect to \(\theta\) and setting the result equal to zero, it is determined that the maximum values of the shear stress occur at \(\theta = \pm 45^\circ\). These planes are illustrated in Figures 11(c) and 11(d), respectively; the stress distribution is found by substituting \(\theta = \pm 45^\circ\) into Eqn. (1.5-5) as follows:

\[
\sigma' = \frac{P}{A_0} \cos^2 45^\circ = \frac{P}{2A_0} \quad \tau_m = \frac{P}{A_0} \sin 45^\circ \cos 45^\circ = \frac{P}{2A_0} \quad \text{(1.5.7)}
\]
It should be noted from Eqn. (1.5-6) that on the planes where the normal stress is maximum, the shear stress is zero; Equation (1.5-7), on the other hand, shows that, in general, the normal stress is not equal to zero on the planes corresponding to maximum shear stress. Consequently, the same loading may produce either, a normal stress and no shear stress; or, a normal and shearing stress of the same magnitude, depending upon the orientation of the section considered.

1.6 Components of Stress

Most structural components are subjected to more involved loading conditions. A different stress (or traction) vector is obtained when different planes are passed through a given point. Figure 12, for example, shows the traction vector resolved into components normal and tangential to the plane under consideration. As mentioned above, the normal component, σ, is called the normal stress, whereas the tangential component, τ, is called the shear stress.

Figure 12. The traction vector can be resolved into normal and tangential components.

A more meaningful description is obtained when one axis of a Cartesian system is aligned with the normal to the plane under consideration. Figure 13, for example, shows the traction vector acting at point P on a plane with its normal, n, aligned with the z-axis.

Figure 13. The traction vector can be resolved into three Cartesian components.
In this case, the tangential component of the traction vector is broken into two in-plane components. A double subscript notation is used to define the stress components; the first subscript denotes the normal to the surface under consideration while the second subscript denotes the direction in which the stress component acts. For the configuration shown in Figure 13, the shear and normal stress components are denoted by \( \tau_{zx}, \tau_{zy}, \sigma_{zz} \) respectively.

In general, three mutually perpendicular planes (with nine stress components) must be passed through a point to completely define the stress state. These nine stress components can then be used to establish the stress distribution on any other plane passed through the point. Figure 14 illustrates that a small cube may be used to facilitate the visualization of the stress condition at a point.

![Diagram](image)

**Figure 14.** The stress distribution referred to an XYZ axes system.

Stresses are shown on three faces of the cube using the subscript notation; the stresses on the back faces with their normals in the negative coordinate directions would be drawn in opposite directions to those shown in the figure. In general, when the outer normal to the plane under consideration is in a positive [negative] coordinate direction, the corresponding stress component acts along a positive [negative] coordinate direction. The stress components corresponding to the planes having their normals aligned with the x and y directions are \( \sigma_{xx}, \tau_{xy}, \tau_{xz} \) and \( \tau_{yx}, \sigma_{yy}, \tau_{yz} \), respectively.

The stress distributions on these three mutually perpendicular planes can be summarized in a stress tensor as follows:

\[
T = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix}.
\]  

The stress state shown in Figure 14 must satisfy equilibrium, namely,
\[ \sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0 \]
\[ \sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0 \quad . \tag{1.6.2} \]

The corresponding force on each face is computed by multiplying the stress by the area. The force summations are satisfied, since stresses on opposite faces cancel. The moment equations require,

\[ \tau_{xy} = \tau_{yx} \quad \tau_{xz} = \tau_{zx} \quad \tau_{yz} = \tau_{zy} \quad . \tag{1.6.3} \]

Hence, six independent components of stress are required to define the condition of stress at a point. Note that, at a given point, shear cannot take place in one plane only; an equal shearing stress must be exerted on another plane perpendicular to the first one.

In conclusion, not only does the stress state change at different points throughout a structural component, but that the same loading condition may lead to different interpretations of the stress situation at a given point by choosing different orientations for planes passed through the point.

### 1.7 Ultimate and Allowable Stress

Knowledge of the stress state is used by engineers in both analysis and design. In general, a structural member must be designed so that its ultimate load is considerably larger than the load that the member will be allowed to carry under normal conditions of utilization. The smaller load is referred to as the allowable load. The ratio of the ultimate load to the allowable load is defined as a factor of safety:

\[ \text{Factor of safety} = F.S. = \frac{\text{ultimaterload}}{\text{allowableload}} \quad . \tag{1.7.1} \]

In many applications, a linear relationship exists between the load and stress caused by the load. In these cases, the factor of safety can be defined in terms of the ultimate stress, \( \sigma_U \), as follows:

\[ \text{Factor of safety} = F.S. = \frac{\text{ultimaterstress}}{\text{allowablestress}} \quad \sigma_U = \frac{P_U}{A} \quad . \tag{1.7.2} \]

The factor of safety is formulated based on variations that may occur in material properties, the number of loadings expected during the life of the part, anticipated future loads, the type of failure, uncertainty in the analysis, and the importance of a given member to overall structural integrity.

**Example:** See Sample Problem 1.3 on page 34 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.
CHAPTER 2 - STRESS AND STRAIN - AXIAL LOADING

2.1 Deformation and Strain under Axial Loading

In addition to the applied stress, deformations caused by applied loads are also important in the analysis and design of structures. Their determination enables forces to be computed that are statically indeterminate within the framework of statics and helps establish the distribution of stress in a structural member.

Figure 1, for example, shows a rod subjected to an axial load P.

![Figure 1](image)

Figure 1. A rod stretches when it is subjected to an axial load.

The normal strain, $\varepsilon$, is given by,

$$
\varepsilon = \frac{\delta}{L}.
$$

(2.1-1)

Since strain is a dimensionless quantity, the same numerical value is obtained when it is computed in U.S. and SI systems.

2.2 Stress-Strain Diagram

A uniaxial tension test can be performed on the specimen shown in Figure 2. In this case, the cross section of the specimen is cylindrical. Two reference points, located at a distance $L_o$ apart, define a gage length. Engineering stress is computed as the load is increased (based on the original cross sectional area) while engineering strain is determined when the elongation experienced by the specimen is divided by the original gage length. A plot of these quantities produces a stress-strain curve, the slope of which provides an important material property (Elastic or Young's modulus).
The shape of the stress-strain curve depends on the material and may change when the specimen is subjected to a temperature change or when the specimen is loaded at a different rate. It is common to classify materials as ductile or brittle. Ductile materials yield at normal temperatures while brittle materials are characterized by the fact that rupture occurs without any noticeable prior change in the rate of elongation. Typical stress-strain curves for such materials are show in Figures 3 and 4.

In the case of a ductile material, the specimen experiences elastic deformation, yields, and strain-hardens until maximum load is reached; necking occurs prior to rupture. Referring to Figure 3, the stress, $\sigma_y$, at which yield is initiated is called the yield stress. The stress, $\sigma_u$, corresponding to the maximum load applied to the specimen is known as the ultimate strength. The stress, $\sigma_B$, corresponding to rupture is defined as the breaking strength.

In the case of the brittle material characterized by Figure 4, there is no difference between the ultimate strength and the breaking strength.

Ductility is measured in terms of percent elongation or percent reduction in area as follows:

$$\% \text{ elongation} = 100 \frac{L_B - L_O}{L_O} \quad \% \text{ reduction in area} = 100 \frac{A_O - A_B}{A_O} \quad (2.2-1)$$

An offset method is used to determine the yield stress in materials where the yield point is not well defined. As illustrated in Figure 5, the yield strength at 0.2% offset is obtained by drawing
through the point of the horizontal axis of abscissa \( \varepsilon = 0.2\% \) (or \( \varepsilon = 0.002 = 2000 \times 10^{-6} = 2000 \) micro-strain), a line parallel to the initial straight-line portion of the stress-strain diagram.

![Stress-strain diagram](image)

\( \sigma (\text{ksi}) \)

\( \varepsilon \) (in/in)

Note: 0.2% = 2000 in/in.

**Figure 5.** An offset method may be used to determine the yield stress of a material.

### 2.3 Hooke's Law; Modulus of Elasticity

In the initial portion of a stress-strain curve,

\[
\sigma = E \varepsilon
\]

(2.3-1)

where \( E \) is defined as the Young's (elastic) modulus of the material. Equation (2.3-1) is valid only for uniaxial tension and is a special case of a generalized set of relations known as Hooke's law (see Section 2.10).

The largest value of stress for which Hooke's law may be used for a given material is known as the **proportional limit**.

### 2.4 Elastic versus Plastic Behavior of a Material; Repeated Loadings-Fatigue

If the strains caused in a test specimen by the application of a given load disappear when the load is removed, the material is said to behave **elastically**. The largest value of stress for which the material behaves elastically is called the **elastic limit** of the material.

A permanent set or **plastic deformation** of the material occurs when the stress exceeds the elastic limit. The stress-dependent part of the plastic deformation is referred to as slip; the time-dependent part is also influenced by temperature and is referred to as **creep**.

A structural component may fail at a level substantially lower than the static breaking strength if it is subjected to repeated loadings. This phenomenon is known as **fatigue** and is evaluated based on a series of experimental tests where specimens are cycled using different maximum stress...
levels. The results of all these tests are plotted as a $\sigma$-n curve; since the number of cycles required for rupture is usually quite large, n is plotted on a logarithmic scale. Typical $\sigma$-n curves are shown in Figure 6.

![Figure 6. Typical $\sigma$-n fatigue plots.](image_url)

The *endurance limit* is defined as the stress for which failure does not occur, even for an indefinitely large number of loading cycles (usually defined as 500 million).

### 2.5 Deformation of Members under Axial Loading

Consider a homogeneous rod BC of length L and uniform cross-sectional area, A, subjected to a centric load $P$. If the resulting axial stress does not exceed the proportional limit of the material, Eqn. (2.3-1) holds and,

$$\varepsilon = \frac{\sigma}{E} = \frac{P}{AE} = \frac{\delta}{L}.$$  \hspace{1cm} (2.5-1)

Hence,

$$\delta = \frac{P}{AE}.$$  \hspace{1cm} (2.5-2)

If the rod is loaded at points other than the end; or, if it consists of several portions of various cross sections and possibly of different materials, it can be divided into component parts and,

$$\delta = \sum \frac{P_i L_i}{A_i E_i}.$$  \hspace{1cm} (2.5-3)

In the case of a rod with variable cross section, or when $P = P(x)$,
When both ends of a rod move, the deformation of the rod is measured by the relative displacement of one end of the rod with respect to the other as follows:

$$\delta_{B/A} = \delta_B - \delta_A = \frac{P L}{AE}.$$  

(2.5-5)


2.6 Statically Indeterminate Problems

There are many problems in which the reactions or internal forces in members cannot be determined from statics alone. These problems are said to be statically indeterminate. To determine either the reactions or internal forces, equilibrium equations must be complemented by relations involving deformations.

It is often found convenient to designate one of the reactions as redundant and to eliminate the corresponding support. The actual solution of the problem is carried out by considering separately the deformations caused by the given loads and by the redundant reaction; then, the results are added. This method is referred to as superposition.

Example: See Examples 2.02-2.05 on pages 78-81 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.

2.7 Problems Involving Temperature Changes

The left portion of Figure 7 shows a homogeneous rod of length L and uniform cross section resting on a smooth horizontal surface. The right portion of the figure shows that the rod expands in all directions as it is subjected to a rise in temperature.

\[ \text{Figure 7. A uniform rod expands in all directions due to an increase in temperature.} \]
In the horizontal direction,

$$\delta_T = \alpha (\Delta T) L$$ \hspace{1cm} (2.7.1)

where $\alpha$ is a constant characteristic of the material called the coefficient of thermal expansion. The latter is expressed in terms of a quantity (strain) per degree change in temperature.

Thus, the thermal deformation is associated with a thermal strain,

$$\varepsilon_T = \frac{\delta_T}{L} = \alpha \Delta T$$ \hspace{1cm} (2.7.2)

Superposition may be applied when considering deformations due to both temperature changes and redundant reactions.

**2.8 Poisson's Ratio**

**Figure 8** shows a **homogeneous** (mechanical properties are independent of the point considered) and **isotropic** (material properties are independent of direction at a given point) bar subjected to axial tension.

![Figure 8](image.png)

**Figure 8.** The axial load causes the bar to elongate in the direction of the load and to contract in the transverse directions.

The stress along the x direction produces an elongation and an axial strain in the direction of the force. The normal stresses on faces perpendicular to the y and z directions are zero; however, contractions occur along these transverse directions which produce lateral strains.

This phenomenon, referred to as the Poisson's effect, is characterized by a material property called the Poisson's ratio. This quantity is defined as,

$$v = - \left[ \begin{array}{c} \text{lateral strain} \\ \text{axial strain} \end{array} \right]$$ \hspace{1cm} (2.8.1)
or

\[
\nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}.
\]  \tag{2.8.2}

And, for uniaxial loading only,

\[
\varepsilon_x = \frac{\sigma_x}{E}, \quad \varepsilon_y = \varepsilon_z = -\frac{\nu \sigma_x}{E}.
\]  \tag{2.8.3}

2.9 Multiaxial Loading

A generalized set of Hooke's laws can be established for an element subjected to stresses along three mutually perpendicular loadings by successively applying Equation (2.8-3) for normal stresses along x, y and z, and then superimposing the results. That is, for the element shown in Figure 9, the components of strain are,

\[
\begin{align*}
\varepsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} - \frac{\nu \sigma_z}{E} \\
\varepsilon_y &= -\frac{\nu \sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu \sigma_z}{E} \\
\varepsilon_z &= -\frac{\nu \sigma_x}{E} - \frac{\nu \sigma_y}{E} + \frac{\sigma_z}{E}.
\end{align*}
\]  \tag{2.9.1}

The results are valid as long as the stresses do not exceed the proportional limit, and as long as the deformations involved remain small. Equation (2.9-1) holds for multiaxial tension but does not take shear stress into account.

Figure 9. An element subjected to three normal stresses only.
2.10 Shearing Strain; Generalized Hooke's Laws

Normal stresses cause extensions or compressions. However, as illustrated in Figure 10, shear stresses may also act on an element. These stresses cause angular distortions.

Figure 10. Shear stresses may also act on an element to produce angular distortions.

A torsion test can be performed on a homogeneous and isotropic material to establish a material property called the shear modulus, $G$. As long as the shearing stresses do not exceed the proportional limit,

$$
\tau_{xy} = G \gamma_{xy} \quad \tau_{yz} = G \gamma_{yz} \quad \tau_{zx} = G \gamma_{zx}.
$$

(2.10.1)

These relations are Hooke's law for shear.

Fortunately, shear stresses have no direct effect on normal strains, and Eqns. (2.9-1) and (2.10-1) can be combined to establish the generalized Hooke's laws for a homogeneous, isotropic material as follows,

$$
\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} - \frac{\nu \sigma_z}{E} \\
\epsilon_y = -\frac{\nu \sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu \sigma_z}{E} \\
\epsilon_z = -\frac{\nu \sigma_x}{E} - \frac{\nu \sigma_y}{E} + \frac{\sigma_z}{E} \\
\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \gamma_{yz} = \frac{\tau_{yz}}{G} \quad \gamma_{zx} = \frac{\tau_{zx}}{G}.
$$

(2.10.2)

In general, these types of equations are called *constitutive* equations and hold on a point wise basis. In particular, the expressions in Eqn. (2.10-2) are referred to as the generalized Hooke's
law for a multiaxial loading. The results are valid as long as stresses do not exceed the proportional limit, and as long as the deformations involved remain small.

It can be shown that,

\[
G = \frac{E}{2(1 + \nu)}.
\]  

(2.10.3)

Consequently, two independent material properties must be established to relate the state of stress to the strain state in a homogeneous and isotropic material.

2.11 Stress and Strain Distribution under Axial Loading; Saint-Venants's Principle

The left hand sketch in Figure 11 shows a specimen subjected to compressive loads that are applied through rigid ends plates. It is reasonable to assume that the member will remain straight, that plane sections will remain plane, and that all elements of the member will deform in the same way (homogeneous strain). However, when concentrated loads are applied as shown in the right hand sketch of Figure 11, the material near the loading points is subjected to large stresses and inhomogeneous strains occur. At points further from the ends, the stresses caused by the concentrated loads distribute throughout the cross sections and a more nearly uniform distribution of stresses and strains results.

Figure 11. At locations away from the applied loads, stresses and strains are more uniform and homogeneous, respectively.

Saint-Venant's principle states that, except in the immediate vicinity of the points of application of the loads, the stress distribution may be assumed independent of the actual mode of application of loads. While the principle makes it possible to replace a given loading by a simpler one for the purpose of computing stresses, the technique cannot be applied to determine stresses at points near the applied loads, and is valid only in cases where the actual loading and the loading used to compute stresses are statically equivalent.
2.12 Stress Concentrations

When a structural member contains a discontinuity, such as a hole or a sudden change in cross section, high localized stresses may occur at the discontinuity. For this reason, one defines a stress concentration factor by the ratio,

\[ K = \frac{\sigma_{\text{max}}}{\sigma_{\text{ave}}} \]  \hspace{1cm} (2.12.1)

Stress concentration factors can be computed in terms of the geometrical factors involved. Figures 12 and 13, for example, show the stress distributions near a circular hole and near fillets in bars subjected to axial loadings.

**Figure 12.** The stress distribution in a tension specimen containing a hole.  
**Figure 13.** The stress distribution in a tension specimen containing fillets.

A photoelastic technique can be used to obtain a plot of stress concentration factors for different geometries. The charts shown in Figure 14 show K-plots for flat bars under axial tension containing holes and fillets.

**Figure 14.** Stress concentration plots for bars containing holes (left) and fillets (right).
Once such a chart has been established, a designer needs only to compute the average stress (based on the area at the discontinuity), and multiply the result obtained by the appropriate value of the stress concentration factor $K$.

### 2.13 Plastic Deformations

For all practical purposes, the elastic limit and yield strength of a material coincide. Consequently, the material behaves elastically until it has reached the yield stress, and regains its original shape after all loads have been removed. To gain some insight into plastic behavior, consider the stress-strain curve shown in Figure 15 for an idealized elastoplastic material.

Figure 15. A stress-strain curve for an idealized elastoplastic material.

Below the yield point, the material behaves elastically and obeys Hooke's law. Once the material reaches the yield point, it keeps deforming under a constant load. When the load is removed, a permanent strain results which can be calculated, since the stress-strain curve corresponding to unloading is parallel to that experienced during loading.

As an example of elastoplastic behavior, consider the bar with a hole shown in Figure 16.

Figure 16. The development of the stress distribution as a bar under tension becomes progressively more plastic.
A plastic zone occurs near the hole when $P$ is increased beyond the load corresponding to yield. As $P$ is increased even further, the stress distribution becomes more uniform across the section.

Residual stresses occur in cases when various parts of a restrained structure undergo different plastic deformations. This may occur as a result of deformations caused by temperature changes. One method of minimizing residual stress is to heat the entire specimen (to 600 degrees Centigrade for steel), and then to cool slowly over a period of 12 to 24 hours.

**Example:** See Sample Problem 2.6 on page 123 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.
CHAPTER 3 - TORSION

3.1 Introduction

Prior discussions were confined to members subjected to axial loads. As illustrated in Figure 1, this chapter deals with members of circular cross section subjected to couples (torques) of magnitude $T$. The member is said to be in torsion.

![Figure 1. A circular shaft subjected to equal and opposite torques is said to be in torsion.](image1)

![Figure 2. When sectioning a shaft, Newton's third law must be satisfied at the interface.](image2)

The most common example of torsional loading is a transmission shaft that is used to transmit power from one point to another.

When analyzing such a situation, Newton's third law must be satisfied. This process is illustrated in Figure 2.

3.2 Stresses in a Shaft

Recall that the normal stresses produced by an axial load were assumed to be uniformly distributed. This is not the case in a shaft in torsion.

![Figure 3 illustrates a section cut perpendicular to the axis of a shaft in torsion. Shear stresses act in this plane and vary linearly with the distance from the axis of the shaft.](image3)

Since the shear stresses cannot take place in one plane only, they must also occur on longitudinal planes. This is illustrated in the three-dimensional element shown as Figure 4.
3.3 Deformations in a Circular Shaft

Figure 5 illustrates that when couples are applied through rigid end plates, every cross section in a circular shaft remains plane and undistorted, and deformations are uniform throughout the shaft. These conditions are not enjoyed, however, by members of noncircular cross section.

Referring to Figure 6, consider a shaft that is attached to a fixed support. When a torque \( T \) is applied to the free end, the shaft twists through an angle, \( \varphi \), called the angle of twist.

The shearing strains are given by,

\[
\gamma = \frac{\rho \varphi}{L}
\]  

\[(3.3-1)\]

Figure 3. A plane is passed through a shaft perpendicular to its longitudinal axis.  
Figure 4. Shear stresses act on this plane and in a perpendicular direction.

Figure 5. Plane sections remain plane only in circular shafts subjected to torsion.  
Figure 6. Torsion causes a circular shaft to rotate through an angle of twist.
where γ and φ are expressed in radians. The shearing strain varies linearly with the distance from the axis of the shaft and reaches a maximum on the outer surface where ρ = c; thus,

\[ \gamma_{\text{max}} = \frac{c \cdot \phi}{L} \quad \text{(3.3-2)} \]

Eliminating φ from Eqns. (3.3-1) and (3.3-2),

\[ \gamma = \frac{\rho}{c} \gamma_{\text{max}} \quad \text{(3.3-3)} \]

3.4 Stresses in the Elastic Range

Hooke's laws hold when all stresses in the shaft remain below the yield strength \( \tau_y \), and

\[ \tau = G \gamma \quad \text{(3.4-1)} \]

or, making use of Eqn. (3.3-3),

\[ \tau = \frac{\rho}{c} \tau_{\text{max}} \quad \text{(3.4-2)} \]

It can also be shown that,

\[ \tau = \frac{T \cdot \rho}{J} \quad \text{(3.4-3)} \]

where \( J \) is the polar moment of inertia. Equation (3.4-3) is known as the elastic torsion formula.

For a solid circular shaft of radius \( c \), and a hollow circular shaft of inner radius \( c_1 \) and outer radius \( c_2 \), the polar moments of inertia are,

\[ J = \frac{1}{2} \pi c^4 \quad J = \frac{1}{2} \pi c_2^4 - \frac{1}{2} \pi c_1^4 = \frac{1}{2} \pi \left( c_2^4 - c_1^4 \right) \quad \text{(3.4-4)} \]

respectively.

From Eqn. (3.4-3), it is evident that the stresses in either a solid or a hollow circular shaft vary linearly with the distance from the center. As illustrated in Figure 7, the maximum value of the shear stress occurs at the outer surface of the shaft.

Equation (3.4-3) can be used for a shaft of variable cross section or for a shaft subjected to torques at locations other than its ends. The value of the internal torque, \( T \), is found by drawing the free-body diagram of the portion of the shaft located on one side of the section. The procedure is illustrated in Figure 8.
Recall that for axial loading, normal stresses act on the planes perpendicular (P/A) and parallel (0) to the longitudinal axis of the member; however, combinations of shear and normal stresses act on other planes. A similar condition holds for torsion. As opposed to the pure shear that acts on the planes perpendicular and parallel to the longitudinal axis, planes which form arbitrary angles experience different combinations of normal and shear stresses. A ductile material subjected to a torsional load will break along the plane perpendicular to the longitudinal axis, since maximum shear stress occurs on these planes. A brittle material tends to break along surfaces that are perpendicular to the direction in which the tensile stress is a maximum; i.e., along surfaces forming a 45 degree angle with respect to the longitudinal axis of the specimen.

The element labeled as 1 in Figure 9 illustrates that the planes of maximum shear in a shaft subjected to torsion are parallel and perpendicular to the longitudinal axis of the shaft. This is one of the few combinations of loading and geometry that produce a zero normal stress on the maximum shear planes. The element labeled as 2 on the figure shows the maximum stress planes on which only normal stresses act.

3.5 Angle of Twist in the Elastic Range

Equations (3.3-1) and (3.4-1) can be combined to derive the following relation between the angle of twist and the applied torque,

\[
\phi = \frac{T L}{J G}
\]  

(3.5-1)

where \( \phi \) is expressed in radians. Equation (3.5-1) is valid within the elastic range and can be used to in conjunction with measurements made in a torsion test to determine the modulus of rigidity (shear modulus) of a given material.

When a shaft is subjected to torques at locations other than its ends, or when it consists of several portions with various cross sections, possibly made of different materials, it can be divided into component parts and,

\[
\phi = \sum_i \frac{T_i L_i}{J_i G_i}
\]  

(3.5-2)

**Figure 10** shows a shaft with a variable cross section. In this case, the angle of twist is given by,

\[
\phi = \int_0^L \frac{T}{J G} \, dx
\]  

(3.5-3)

![Diagram of a shaft with variable cross section](image)

**Figure 10.** An integral formulation must be used for a shaft with variable cross section.
The angle of twist can be calculated for a shaft in which the ends move relative to one another by applying,

\[ \phi_{A/B} = \phi_A - \phi_B = \frac{T}{J} \frac{L}{G} \]  

(3.5-4)

**Example:** See Sample Problem 3.4 on page 166 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.

### 3.6 Statically Indeterminate Shafts

When the internal torque cannot be calculated from statics alone, equilibrium equations must be complemented by relations involving the deformations of the shaft that are obtained by considering the geometry of the problem.

### 3.7 Design of Transmission Shafts

In choosing a transmission shaft, a designer must select the material and the dimensions of the cross section of the shaft so that the maximum shearing stress allowable in the material will not be exceeded when the shaft is transmitting the required power at a specified speed. The power, \( P \), associated with the rotation of a rigid body subjected to torque, \( T \), is

\[ P = T \omega \]  

(3.7-1)

where \( \omega \) is the angular velocity (rad/sec). Since \( \omega = 2\pi f \),

\[ P = 2\pi f T \]  

(3.7-2)

where \( f \) is the frequency expressed in hertz (Hz). For design purposes, Eqn. (3.7-2) may be used in conjunction with the elastic torsion formula, Eqn. (3.4-3).

If SI units are used, with \( f \) expressed in Hz and \( T \) in Nm, the power will be expressed in Nm/sec (watts; W). When U.S. customary units are used, the frequency is usually expressed in rpm and the power in horsepower (hp). It is then necessary, before applying Eqn. (3.7-2), to convert the frequency into revolutions per second (Hz) and the power to ft lb/sec or in lb/sec. Note,

\[ 1 \text{ rpm} = \frac{1}{60} \text{ s}^{-1} = \frac{1}{60} \text{ Hz} \]  

(3.7-3)

\[ 1 \text{ hp} = 550 \text{ ft} \cdot \text{lb} / \text{s} = 6600 \text{ in} \cdot \text{lb} / \text{s} \]  

In addition,

\[ 1 \text{ ft} \cdot \text{lb} / \text{sec} = 1.356 \text{ W} \quad 1 \text{ hp} = 745.7 \text{ W} \]  

(3.7-4)
3.8 Stress Concentrations in Circular Shafts

In practice, torques are applied through flange couplings or through gears, and the distribution of stresses, in and near the section where torques are applied, is usually very different from that given by the torsion formula. Furthermore, in the case of a shaft with an abrupt change in the diameter of its cross section, the torsion formula may be applied, but the stress concentration is often taken into account. In Figure 11, for example, the highest stress occurs at interface between the different diameter shafts.

![Diagram of a fillet reducing stress concentration](image)

Figure 11. A fillet may be used to reduce the stress concentration that occurs between shafts having different diameters.

These stresses may be reduced through the use of a fillet with radius, r; the maximum shearing stress at the fillet may be expressed as

\[ \tau_{\text{max}} = K \frac{T_c}{J} \]  

(3.8-1)

where \( T_c/J \) is the stress computed for the smaller-diameter shaft, and \( K \) is the stress concentration factor. Equation (3.8-1) holds as long as the maximum shear stress does not exceed the proportional limit of the material. If plastic deformations occur, they will produce a maximum stress lower than that indicated by Eqn. (3.8-1). Figure 12 shows stress-concentrations for fillets in circular shafts.

3.9 Torsion of Noncircular Members

The validity of the assumption that the cross section of a circular member subjected to torsion remains plane and undistorted depends upon the axis symmetry of the member. In members of noncircular cross section, a cross section will be warped and the stress distribution is not governed by the equations presented in this chapter. For example, in a bar with a square cross section, the shearing stress is zero at the corners of the cross section, and is maximum along the center line of each of the faces of the bar.
As illustrated in Figure 13, a membrane analogy may be used to visualize the shearing stress in any straight bar of uniform, noncircular cross section. In this test, a homogeneous elastic member is attached to a fixed frame and subjected to a uniform pressure from below. The shearing stresses will have the same direction as the horizontal tangent to the membrane, with magnitudes proportional to the slope of the membrane. In addition, the applied torque will be proportional to the volume between the membrane and the plane of the fixed frame.
CHAPTER 4 - PURE BENDING

4.1 Stresses and Deformations in Symmetric Members in Pure Bending

As illustrated in Figure 1, a member subjected to equal and opposite couples acting in the same longitudinal plane is said to be in pure bending. Figure 2, for example, shows a prismatic member possessing a plane of symmetry, subjected at its ends to equal and opposite couples acting in the plane of symmetry. Since the bending moment is the same in any cross section, the member remains symmetric with respect to the plane of symmetry and bends uniformly. Lines AB and A'B', that were originally straight, are transformed into circular arcs. When stresses remain in the elastic range,

\[
\frac{1}{\rho} = \frac{M}{E I}
\]  

(4.1-1)

where E is the elastic modulus, and I is the moment of inertia measured with respect to the centroidal axis parallel to the Z axis.

At any point of the member, a state of uniaxial stress exists. This is evident in Figure 2 where line AB decreases in length (compression) and A'B' becomes longer (tension). It follows that there is a surface parallel to the lower and upper faces of the member in which \(\varepsilon_x\) and \(\sigma_x\) are zero. This "neutral" surface intersects the plane of symmetry in a line called the neutral axis. For a beam subjected to pure bending with stresses in the elastic range, the neutral axis passes through the centroid of the cross section.

**Figure 1.** Pure bending occurs when equal and opposite couples are applied as shown.  
**Figure 2.** Longitudinal elements deform into circular arcs when subjected to pure bending.
It can be shown that the longitudinal strain $\varepsilon_x$ varies linearly throughout the member with the distance $y$ from the neutral surface. That is,

$$\varepsilon_x = -\frac{y}{\rho} \quad (4.1-2)$$

where $\rho$ is the radius of curvature. The maximum strain occurs at the largest distance from the neutral axis (at the upper or lower surface of the member). Denoting the absolute value by $\varepsilon_m$,

$$\varepsilon_m = \frac{c}{\rho} \quad \text{and} \quad \varepsilon_x = -\frac{y}{c} \varepsilon_m \quad (4.1-3)$$

Since the normal strain is linearly proportional to the normal stress in a tension member,

$$\sigma_x = -\frac{y}{c} \sigma_m \quad (4.1-4)$$

The normal stress and the maximum absolute normal stress, $\sigma_m$, in the beam are given by,

$$\sigma_x = -\frac{M}{I} \quad \text{and} \quad \sigma_m = \frac{M}{I} \quad (4.1-5)$$

respectively. A plot of the stress distribution over the cross section is shown in Figure 3.

**Figure 3.** In pure bending, the normal stress varies linearly through the thickness.

Equation (4.1-5) is the elastic flexure formula, and the normal stress due to bending is often referred to as the flexural stress. The ratio $I/c$ depends only on the geometry of the cross section and is referred to as the elastic section modulus, $S$. Consequently,
\[
S = \frac{I}{c} \quad \text{and} \quad \sigma_m = \frac{M}{S} .
\] (4.1-6)

In general, beams are designed with as large a value of \( S \) as practicable. Figure 4, for example, shows the cross sections of two wooden beams having rectangular cross sections of width \( b \) and depth \( h \); note that the areas of the two cross sections are equal. In this case,

\[
S = \frac{I}{c} = \frac{1}{12} \frac{b h^3}{h/2} = \frac{1}{6} b h^2 = \frac{1}{6} A h
\] (4.1-7)

where \( A \) is the cross-sectional area of the beam. It is evident that the beam with the larger depth, \( h \), will have the larger section modulus; thereby, making it more effective in resisting bending moments.

Figure 5 illustrates that in the case of structural steel, American standard beams (S-beams) and wide-flange beams (W-beams) are preferred to other shapes because a large portion of their cross sections is located far from the neutral axis. This provides a large value of \( I \); and, consequently, of \( S \).

![Figure 4](image)

**Figure 4.** Different cross sections provide different moments of inertia.

**Figure 5.** “I” beams are often used in construction because of their large \( I \) values.

### 4.2 Deformations in a Transverse Cross Section

Elements in a beam subjected to pure bending are in uniaxial stress. In addition to deformations along the axis of the member (\( x \)), elements are deformed in the transverse (\( y \) and \( z \)) directions. The normal strains depend on the Poisson's ratio and can be expressed as,

\[
\varepsilon_y = -\nu \varepsilon_x \quad \varepsilon_z = -\nu \varepsilon_x
\] (4.2-1)

or, recalling Eqn. (4.1-2),

4.3
\[ \varepsilon_y = \frac{V y}{\rho} \quad \varepsilon_z = \frac{V y}{\rho} . \]  

(4.2-2)

In this case, the neutral axis of the transverse section is bent into the radius of a circle. As illustrated in Figure 6, the center of this circle is located on the side opposite from the center of curvature of the member. The reciprocal of the corresponding radius of curvature, \(1/\rho'\), represents the curvature of the transverse cross section and is called the anticlastic curvature. The latter is related to the curvature, \(1/\rho\), by,

\[ \frac{1}{\rho'} = \frac{v}{\rho} \]  

(4.2-3)

\[ \rho' = \rho/v \]

Figure 6. The transverse section is also bent into a circle.

**Example:** See Sample Problem 4.2 on page 236 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.

### 4.3 Bending of Members Made of Several Materials

In many cases, the cross section of a beam subjected to pure bending is made up of two or more materials with different modula of elasticity. The normal strain still varies linearly with the distance \(y\) from the neutral axis of the section and

\[ \varepsilon_x = -\frac{y}{\rho} . \]  

(4.3-1)

4.4
The method of attack for solving for the corresponding stress distribution is to formulate a transformed section. This is accomplished by choosing one of the materials (with $E_1$) as a standard, and then formulating a nondimensional ratio in sections made from different materials as follows:

$$n = \frac{E_2}{E_1}.$$  \hspace{1cm} (4.3-2)

As illustrated in Figure 7, the transformed section is obtained by stretching each element in a direction parallel to the neutral axis of the section (by multiplying that dimension by $n$), and stresses are computed, assuming that the section is made from a homogeneous material, using

$$\sigma_x = -\frac{nM_y}{I}$$

where $y$ is the distance from the neutral surface, and $I$ is the moment of inertia of the transformed section with respect to its centroidal axis. The value of $n$ for the material selected as the standard is 1.0.

Figure 7. A transformed section is constructed for beams made from different materials.

Again, the stresses calculated for the material with $E_1$ are those obtained from Eqn. (4.3-3) using $n = 1.0$; stresses in other portions of the structure are obtained by multiplying the stresses in the transformed section by $n$.

The curvature of the composite member can be calculated using

$$\frac{1}{\rho} = \frac{M}{E_1 I}.$$  \hspace{1cm} (4.3-4)
An important example of structural members made of two materials is furnished by reinforced concrete beams. The method of attack is slightly different in that only the portion of the cross section located above the neutral axis is used in computing stresses in the transformed section (see pages 245 and 249 in the 6th edition of Beer, Johnston, DeWolf, and Mazurek for details).


4.4 Stress Concentrations

Pure bending results only when couples are applied through rigid end plates. Under other loading conditions, stress concentrations will exist near the points where loads are applied.

Higher stresses will also occur if the cross section of the member undergoes a sudden change in geometry. Two particular cases of interest have been studied, the case of a flat bar with a sudden change in width, and the case of a flat bar with grooves. The stress concentration factors can be found from the data included in Figure 8.

Figure 8. Stress concentration plots for bending specimens containing fillets and grooves.

The maximum value of stress that occurs in the critical cross section may be expressed as

$$\sigma_m = K \frac{M c}{I}$$

(4.4-1)

where K in the stress concentration factor, and where c and I refer to the critical section; that is, to the section of width d in both of the cases considered above.
4.5 Eccentric Axial Loading in a Plane of Symmetry

When the line of action of loads applied to a member does not pass through the centroid of the cross section, the member is subjected to eccentric loading. The following analysis will be limited to members that possess a plane of symmetry, and that the loads are applied in the plane of symmetry. An example is shown in Figure 9.

![Figure 9. A member subjected to eccentric axial loading in a plane of symmetry.](image)

In this case, the internal forces acting on the member can be represented by a force, $F$, applied at the centroid of the section and a couple, $M$, acting in the plane of symmetry.

The stress distribution is obtained by superimposing the uniform stress distribution corresponding to the applied loads and the linear distribution corresponding to the bending couples. Recalling Eqns. (1.2-1) and (4.1-5)

\[
\sigma_x = \frac{P}{A} = \frac{M y}{I}.
\]  

Equation (4.5-1) shows that the distribution of stresses across the section is linear but not uniform. Results are valid to the extent that the conditions of applicability of the superposition principle and Saint Venant's principle are met. Depending upon the geometry of the cross section and the distance, $d$ (see Fig. 9), the combined stresses may all have the same sign, or some may be positive and others negative. Two different examples are illustrated in Figure 10.

**Example:** See Sample Problem 4.8 on page 273 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.
4.6 Unsymmetric Bending

The analysis of pure bending has been limited to members possessing at least one plane of symmetry and subjected to couples acting in that plane. In these cases, the neutral axis of the cross section coincides with the axis of the couple. When bending couples do not act in the plane of symmetry, or when the member does not possess a plane of symmetry, the directions of the neutral axis and the axis of the couple do not coincide. One of these conditions is illustrated in Figure 11.

![Diagram](image)

**Figure 10.** The stresses produced by eccentric bending are obtained by superimposing the stresses caused by the applied loads and the bending couples.

**Figure 11.** In some cases, the neutral axis of the section does not coincide with the axis of the bending couple.
The exceptions, illustrated in the upper portion of the figure, occur when the couple vector is directed along one of the (two) centroidal axes of the cross section (where the products of inertia vanish). These directions may be obtained using transformation equations graphically represented by Mohr's circle (see Chapter 6).

The principle of superposition may be used to solve for stresses in the most general case of unsymmetrical bending. As illustrated in Figure 12, centroidal axes are first established (using transformation equations or by inspection).

![Image of unsymmetrical bending](image)

**Figure 12.** The method of superposition is applied to solve for the stresses in unsymmetrical bending.

Then, the moment vector is resolved into two components along the centroidal axes (x and y). The resulting stress distribution is calculated using,

\[
\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}.
\]  

Equation (4.6-1) shows that the stress distribution is linear. In general, however, the neutral axis of the cross section does not coincide with the axis of the bending couple. The angle \(\phi\) that the neutral axis forms with the z axis is defined by the relation,

\[
\tan \phi = \frac{I_z}{I_y} \tan \theta
\]

where \(\theta\) is the angle between the couple vector, \(\mathbf{M}\), and the z-axis.
4.7 General Case of Eccentric Axial Loading

The stresses produced in a member by an axial eccentric load applied in a plane of symmetry have been studied in Section 4.5. As illustrated in Figure 13, when an axial load is not applied in a plane of symmetry, the load can be moved to the centroid.

![Figure 13. Eccentric axial loading.](image)

The equivalent loading will consist of a force-couple system. The moment vector may be resolved along the principal centroidal axes, and stresses are obtained by superimposing the stresses due to unsymmetrical bending with those due to the centric load.

Mathematically, the stresses are computed using,

\[
\sigma_x = \frac{P}{A} - \frac{M_z}{I_z} y + \frac{M_y}{I_y} z.
\]  \hspace{1cm} (4.7-1)

Equation (4.7-1) shows that the distribution of stresses across the section is linear. In computing the combined stress, care should be taken to determine the sign of each of the three terms in the right-hand side of the equation, since each of these terms may be positive or negative.

**Example:** See Sample Problem 4.9 on page 287 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.
CHAPTER 5 - ANALYSIS AND DESIGN OF BEAMS FOR BENDING

5.1 Shear and Bending Moment in a Beam

A beam is a structural member that is designed to support loads at various points along its span. In most cases, loads are perpendicular to the beam. Recall that the transverse loading of a prismatic beam may cause both normal and shearing stresses on any given transverse section of the beam. Within the elastic range, the largest value of bending stress will occur in the section where the magnitude of the moment is maximum and, as described previously in Section 4.1,

\[ \sigma_m = \frac{|M|_{\text{max}} c}{I} \]  \hspace{1cm} (5.1-1)

In most common types of beams (rectangular beams, S-beams, W-beams), the shearing stress on a given section is maximum at the neutral axis, and occurs where the magnitude of the shear force is maximum. As described later in Section 6.1,

\[ \tau_m = \frac{|Q|_{\text{max}} Q}{I t} \]  \hspace{1cm} (5.1-2)

The goal of this section is to determine the shear forces and bending moments over the span, with the overall goal of selecting the material and shape and dimensions of the cross section to resist these shearing forces and bending moments. The design of a prismatic beam is covered later in Section 8.2.

These computations will be considerably simplified if the values of the shear \( V \) and of the bending moment \( M \) in the various sections of the beam are plotted against the distance \( x \) measured from one end of the beam. The graphs obtained in this way are called the shear diagram and bending-moment diagrams, respectively.

The method of attack is to determine the reactions at points of constraint, then cut the beam at various sections along the span, indicating on which portion of the beam internal forces are acting. As illustrated in Figure 1, \( V \) and \( M \) are determined by considering the equilibrium in the portion of the beam located on either side of the section considered.

A convention must be established; the shear \( V \) and the bending moment \( M \) at a given point of a beam are assumed to be positive when the internal forces and couples acting on each element of the beam are directed as shown in Figure 2.
Example:  Draw the shear and moment diagram for the beam shown in Figure 3.

The first step in the solution is to determine the reactions at the points of constraint by drawing an equivalent FBD shown in Figure 4. Once this calculation is performed disregard the equivalent beam and return to the problem at hand. Do not draw the shear and bending moment diagrams for the equivalent beam. Referring to Figure 4:

\[ +\sum M_B = 0 = 4(20) - 10 R_C + 12(8) \quad R_C = 17.6 \text{kips} \]
\[ +\sum F_x = 0 = R_{Bx} \quad R_{Bx} = 0 \text{kips} \]
\[ +\sum F_y = 0 = -8 + R_C - 20 + R_{By} \quad R_{By} = 10.4 \text{kips} \]

Figure 1. The shear and moment are established at a section by cutting the beam.  

Figure 2. The sign convention used for constructing shear and moment diagrams. 

Figure 3. A beam is subject to a distributed load between A and C and a concentrated load applied at D. 

Figure 4. An equivalent FBD corresponding to the beam shown in Fig. 3.
The beam must be sectioned each time there is a discontinuity in loading. **Figures 5 through 7** show the FBDs for cuts made between points A and C, C and D, and D and B, respectively. Once the cut has been made, an equivalent system can be constructed to determine V and M. Note that the opposite convention is required when working with the right hand portion of the beam (see **Figure 7**).

![Figure 5](image1)

**Figure 5.** FBD for the cut between A and C.

![Figure 6](image2)

**Figure 6.** FBD for the cut between C and D.

Referring to **Figure 5**:

\[
\begin{align*}
+ \uparrow F_y &= 0 = -2x - V \\
V &= -2x
\end{align*}
\]

\[\text{(2)}\]

\[
+ \overset{\circ}{}\sum M_o = 0 = M + 2x \left( \frac{1}{2} x \right)
\]

\[M = -x^2.\]

Referring to **Figure 6**:

\[
\begin{align*}
+ \uparrow F_y &= -8 + 17.6 - V \\
V &= 9.6
\end{align*}
\]

\[\text{(3)}\]

\[
+ \overset{\circ}{}\sum M_o = 0 = M + 8 \left( x - 2 \right) - 17.6 \left( x - 4 \right)
\]

\[M = -54.4 + 9.6x.\]

Referring to **Figure 7**:

\[
\begin{align*}
+ \uparrow F_y &= 0 = V + 10.4 \\
V &= -10.4
\end{align*}
\]

\[\text{(4)}\]

\[
+ \overset{\circ}{}\sum M_o = 0 = -M + 10.4 \left( 14 - x \right)
\]

\[M = 145.6 - 10.4x.\]

5.3
Figure 8 shows the results when the shear force and bending moment are plotted over the span of the beam.

5.2 Relations among Load, Shear, and Bending Moment

Figure 9 shows a beam subjected to a distributed loading. A portion of the span has been isolated for consideration.

Figure 9. A beam subjected to a distributed loading.

It can be shown that the distributed loading is related to the slope of the V versus x curve as follows:
Integrating Equation (5.2-1) between points C and D,

\[ V_D - V_C = - A_{CD} \]  

where \( A_{CD} \) is the area under the load curve between points C and D.

Equation (5.2-1) is not valid at a point where a concentrated load is applied; the shear is discontinuous at such a point. Similarly, Eqn. (5.2-2) ceases to be valid when concentrated loads are applied between C and D, and the equation should be applied only between successive concentrated loads.

In addition,

\[ \frac{dM}{dx} = V \]  

Equation (5.2-3) indicates that the slope of the bending moment curve is equal to the value of the shear, and

\[ M_D - M_C = A_{CD} \]  

where \( A_{CD} \) is the area under the shear curve between points A and C.

The area under the shear curve should be considered positive where the shear is positive and negative where the shear is negative. Equation (5.2-4) is valid even when a concentrated load is applied between points C and D, but can not be applied when a couple is applied at a point in that region. An applied couple results in a jump discontinuity in the moment curve with counterclockwise moments forcing the curve downward; clockwise moments force it upward.

Note that if the load curve is uniform, the shear curve is an oblique straight line (first degree), and the bending moment curve is a parabola (second degree). That is, the shear and bending moment curves will always be, respectively, one and two degrees higher than the load curve. This information helps one to sketch the diagrams, once a few values of shear and bending moment have been computed.

CHAPTER 6 - SHEARING STRESSES IN BEAMS AND THIN-WALLED MEMBERS

6.1 Determination of Shearing Stresses in a Beam

This chapter presents the analysis of stresses in prismatic members subjected to transverse loads. Figure 1, for example, shows a cantilever beam subjected to a single upward force, P. Assume that the beam possesses a vertical, longitudinal plane of symmetry, and that P is applied in that plane. The internal forces consist of a bending moment M which varies with x, and a vertical shear force V. Fortunately, the shear force V causes deformations but these deformations do not affect the normal stresses caused by M. Hence, the resultant stress distribution is the superposition of the stresses resulting from the two different loadings.

As illustrated in Figure 2, the shear force produces a shear stress, $\tau_{xy}$ that represents both the vertical component of the shearing stress on a section perpendicular to the axis of the beam, and the longitudinal component of the shearing stress on a horizontal section. The bending moments produce a normal stress that varies linearly over the cross section.
Referring to Figure 3, a horizontal shearing force acts on A'C'.

![Diagram of a horizontal shearing force on a cross-section of a beam]

**Figure 3.** The horizontal shearing force gives rise to a shear flow.

This leads to the definition of a quantity called the shear flow (shear per unit length) defined by,

\[ q = \frac{VQ}{I} \tag{6.1-1} \]

where \( I \) is the moment of inertia of the entire cross section, and \( Q \) is the first moment of the area about the neutral axis of the portion of the cross section that is located either above or below \( C' \). The latter is computed by multiplying the appropriate area by the distance to its centroid.

The average shear stress can be computed by considering the cut shown in Figure 3. Since,

\[ \Delta H = q \Delta x = \frac{VQ}{I} \Delta x \quad \Delta A = t \Delta x \quad \tau_{ave} = \frac{\Delta H}{\Delta A} = \frac{VQ}{I} \frac{\Delta x}{t \Delta x} \tag{6.1-2} \]

the average shear stress is given by

\[ \tau_{ave} = \frac{VQ}{It} \tag{6.1-3} \]

where \( t \) is the width of a cut made through the point in question, perpendicular to the shear force.

Note that, while \( Q \) is maximum for \( y = 0 \), we cannot conclude that the average stress will be maximum along the neutral axis, since \( \tau_{ave} \) depends upon the width of the section. As long as the width of the beam cross section remains small, compared to its depth, the shearing stress varies only slightly over the cut. This is illustrated in Figure 4.
6.2 Discussion of Stresses in a Narrow Rectangular Beam

In a narrow rectangular beam (b < 1/4 h) the variation of shearing stress across the width of the beam is less than 0.8% of the average value.

Figure 5 illustrates that, for a beam of rectangular cross section, the distribution of shearing stresses in a transverse section is parabolic with a maximum shear stress, equal to 3/2 (V/A), occurring at the neutral axis. As illustrated in Figure 6, the stress state is significantly more complex in other sections.

For Eqn. (6.1-3) to be valid, P must be distributed parabolically. This situation is rarely encountered in practice and stresses can not be determined using the equation for points close to the point of application of the load. However, Saint-Venant’s principle predicts that stresses can be calculated in other locations. In the more general case, when a beam is subjected to several concentrated forces, the principle of superposition may be used to determine the normal and shear stresses in sections located between the points of application of the loads. In portions of the beam located under a concentrated load, normal stresses will be exerted on the horizontal faces of a cubic element, and the analysis becomes more complicated.

6.3 Shearing Stresses on an Arbitrary Longitudinal Cut - Thin Walled Members

When a beam is subjected to a transverse loading in its plane of symmetry, the horizontal shearing force per unit length or shear flow, q, exerted on an arbitrary longitudinal cut may be obtained from Eqn. (6.1-1). This is illustrated by Figure 7 in which the average shearing stress along the cut is given by Eqn. (6.1-3).

These observations may be used to determine the shear stresses in thin walled members. Consider, for example, the “I” beam shown in Figure 8.

Figure 7. The shear flow and shearing stress may be computed for a longitudinal cut.

Figure 8. The concept of shear flow can be applied to thin walled members.

6.4
In each case, a cut has been made perpendicular to the surface of the member and Eqn. (6.1-3) gives the shearing stress in the direction tangent to that surface.

It should be noted that, in thin walled members; the other component of the shearing stress may be assumed to be zero in view of the proximity of the two free surfaces. Similar arguments may be applied to determine shear stresses in other thin walled shapes such as those illustrated in Figure 9.

![Figure 9](image1.png) **Figure 9.** The shear stress perpendicular to the free surfaces is neglected.

Since V and I are constant in any given section, the shear flow, q, depends only upon the first moment Q and thus may be sketched on the section as demonstrated in Figure 10. The sense of q in the horizontal portions of the section may be easily obtained from its sense in the vertical portions (which is the same as the sense of the shear V).

![Figure 10](image2.png) **Figure 10.** The sense of the shear flow can be obtained by inspection.

**Example:** See Sample Problem 6.3 on page 406 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.
CHAPTER 7 - TRANSFORMATIONS OF STRESS AND STRAIN

7.1 Introduction

In Chapter 1, it was shown that for every point in a loaded member, the components of shear and normal stress change as different planes are passed through the point. Even though an infinite number of planes can be passed through the point, it is sufficient to determine only the stresses on three mutually perpendicular planes to completely characterize the state of stress on an arbitrary plane. There are, however, certain critical planes, the analysis of which help to determine the structural integrity. This chapter considers the transformation equations required to pinpoint the orientations of these critical planes and the corresponding stresses that act on them.

7.2 Stress Transformation Equations

Figure 1 shows the distribution of stress on planes with normals in the x,y,z directions while Figure 2 illustrates that both normal and shear stresses change under a rotation of coordinate axes.

![Figure 1. The stress distribution referred to an XYZ axes system.](image1)

![Figure 2. The stress distribution referred to a rotated X'Y'Z' axes system.](image2)

We will begin our analysis by considering a point in plane (biaxial or 2-D) stress, in which two faces of the cubic element are free of any stress. This situation is encountered on the surface of an element or component that is not subjected to an external force. Figure 3 shows the stress distribution on planes with their normals in the X and Y directions; Figure 4, on the other hand,
shows the distribution of stress on the faces of an element rotated counterclockwise around the Z axis by an angle $\theta$.

Figure 3. The stress distribution on a plane stress element referred to XY axes. Figure 4. The stress distribution on a plane stress element referred to X'Y' axes.

The stresses in the rotated axes system may be expressed in terms of the original system as follows,

\[
\begin{align*}
\sigma'_{xx} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\
\sigma'_{yy} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \\
\tau'_{xy} &= -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta .
\end{align*}
\] (7.2-1)

7.3 Mohr's Circle

The expressions in Eqn. (7.2.1) can be manipulated to yield,

\[
\left(\sigma_{xx} - \frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2 .
\] (7.3-1)

Equation (7.3-1) is the equation of a circle and stresses can be graphically represented as shown in Figure 5. The circle is drawn by plotting points X with coordinates $\sigma_x$ and $-\tau_{xy}$, and Y with coordinates $\sigma_y$ and $\tau_{xy}$ . Normal stresses are assumed positive when they act away from the element; positive shear stresses produce a counterclockwise rotation of the element. The center of the circle lies at:
\[ \sigma_c = \frac{\sigma_{xx} + \sigma_{yy}}{2} \quad \tau_c = 0 \quad (7.3-2) \]

and the radius of the circle is:

\[ R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \quad (7.3-3) \]

Each radial line in the circle represents one of the infinite number of planes that can be passed through the point and the intersection of the line with the circle defines the shear and normal stresses acting on that plane. The orientation of the planes are usually measured with respect to the X axis and are determined by computing angles on the circle and dividing these values in half. The resulting angle is measured in the same sense (clockwise or counterclockwise) on the element.

Points A and B are of special interest, since they correspond to the principal planes on which the principal stresses act. Note that the shear stress is zero for these points. Other points of interest lie at D and E where the maximum in-plane shearing stress occurs. This value is the absolute maximum shearing stress when the principal normal stresses are of opposite sign. Note that the normal stress on these planes corresponds to the location of the center of the circle.

The values of normal and shear stress at these critical locations can be computed graphically, as demonstrated in the example problems that follow this section, or calculated on the basis of Eqn. (7.2-1) as described below.
The orientation of the principal planes with respect to the X axis, \( \theta_p \), are obtained by differentiating the first expression in Eqn. (7.2-1) with respect to \( \theta \) and setting the result equal to zero. This operation produces

\[
\tan 2 \theta_p = \frac{2 \tau_{xy}}{\sigma_{xx} - \sigma_{yy}}. \tag{7.3-4}
\]

Equation (7.3-4) produces two values of \( \theta_p \), 90\(^0\) apart; one corresponds to the maximum and the other to the minimum value of normal stress.

Expressions for the magnitudes of these principal stresses and the corresponding shear stresses can be found by substituting the values of \( \theta_p \) found from Eqn. (7.3-4) back into the expressions in Eqn. (7.2-1). This operation produces:

\[
\sigma_{\text{max},\text{min}} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}, \tag{7.3-5}
\]

The principal stress (maximum or minimum) that corresponds to a particular value of \( \theta_p \), is the one that results when the latter is substituted into the first expression in Eqn. (7.2-1).

A similar argument can be applied to the third expression in Eqn. (7.2-1). That is, the planes of maximum in-plane shear stress are oriented at \( \theta_s \) with respect to the X axis, where

\[
\tan 2 \theta_s = -\frac{\sigma_{xx} - \sigma_{yy}}{2 \tau_{xy}} \tag{7.3-6}
\]

and

\[
\tau_{\text{max}} = \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \quad \text{and} \quad \sigma' = \sigma_c = \frac{\sigma_{xx} + \sigma_{yy}}{2}. \tag{7.3-7}
\]

Note that the principal planes are oriented at an angle of 45\(^0\) with respect to the planes of maximum shear. The normal stress on the planes of maximum and minimum shear is equal but not necessarily zero.

The shear (maximum or minimum) that corresponds to a particular value of \( \theta_s \), is the one that results when the latter is substituted into the third expression in Eqn. (7.2-1).

**Example:** See Sample Problem 7.2 on page 457 in the 6\(^{th}\) edition of Beer, Johnson, DeWolf, and Mazurek.
7.4 General State of Stress

In general, the stress tensor consists of nine components of stress, six of which are independent. The nine values correspond to three normal and six shear stress components acting on three mutually perpendicular planes. As illustrated by Figure 6, the coordinate axes can be rotated, in much the same manner as in plane stress, to determine principal planes on which the shear stress is zero.

![Figure 6. A rotated element showing the three principal planes.](image)

In mathematical terms, this is called an eigenvalue problem. One obtains a characteristic equation that has three roots called eigenvalues; each eigenvalue is equal in magnitude to one of the principal stresses. Eigenvectors are determined for each eigenvalue, and these vectors define the orientation of the normal to each of the principal planes on which the principal stresses act.

Figure 7 shows one possible configuration for graphically depicting the stress distribution at a point.

![Figure 7. The Mohr's circle for an arbitrary point at which the three eigenvalues are all positive.](image)
In this case, the three eigenvalues are all positive. Coordinate rotations around the three eigenvectors are pictorially represented by the three circles, and points on their circumferences define the stress distribution on different planes on a rotated element. A more general coordinate transformation leads to a point inside of the area between the inner circles and the outer circle.

In all cases,

$$\tau_{\text{max}} = \frac{1}{2} | \sigma_{\text{max}} - \sigma_{\text{min}} |$$  \hspace{1cm} (7.4-1)

Two cases are of interest when evaluating Eqn. (7.2-1) for the case of plane stress (when one of the eigenvalues is zero). Figure 8 shows that when the principal stresses obtained from the bi-axial transformation (the 2-D Mohr's circle shown as the solid line in the figure) are of opposite sign, the in-plane maximum shear stress is the absolute maximum shear stress at the point under study.

Figure 8. The principal stresses are of opposite sign.

As illustrated in Figures 9 and 10, the corresponding planes are oriented at 45 degrees with respect to the principal directions.
Figure 11, on the other hand, illustrates that when the in-plane principal stresses are of the same sign (see the solid circle), the maximum shear stress (shown at D') must be calculated based on the knowledge that the third eigenvalue is zero; i.e., equal to one-half the stress at A.

Figure 11. The principal stresses are of the same sign.

The orientations of the maximum shear stress planes are shown in Figures 12 and 13.

Figure 12. Orientation of the maximum shear stress planes.

Figure 13. Orientation of the maximum shear stress planes.

7.5 Stresses in Thin-Walled Pressure Vessels

A thin walled pressure vessel provides an important application of the analysis of plane stress. The following discussion will be confined to cylindrical and spherical vessels.
The cylindrical vessel shown in Figure 14 has *inside* radius \( r \) and wall thickness \( t \). The vessel contains a fluid under pressure, \( p \).

![Figure 14. A cylindrical pressure vessel.](image)

The hoop stress, \( \sigma_1 \), and the longitudinal stress, \( \sigma_2 \), are given by:

\[
\begin{align*}
\sigma_1 &= \frac{p r}{t} \\
\sigma_2 &= \frac{p r}{2 t}
\end{align*}
\]  
(7.5-1)

The Mohr's circle corresponding to the cylindrical pressure vessel is shown in Figure 15.

![Figure 15. Mohr's circle for a cylindrical pressure vessel.](image)

Note,

\[
\tau_{\text{max (in plane)}} = \frac{1}{2} \sigma_2 = \frac{p r}{4 t}
\]  
(7.5-2)

And

\[
\tau_{\text{max}} = \sigma_2 = \frac{p r}{2 t}
\]  
(7.5-3)

7.8
Figure 16, on the other hand shows a spherical pressure vessel of inside radius $r$ and wall thickness $t$, containing a fluid of gage pressure $p$.

![Figure 16. A spherical pressure vessel.](image)

In this case,

$$
\sigma_1 = \sigma_2 = \frac{p \cdot r}{2 \cdot t}.
$$

(7.5-4)

The Mohr’s circle for the spherical pressure vessel is shown in Figure 17.

![Figure 17. Mohr’s circle for a spherical pressure vessel.](image)

Note,

$$
\tau_{\text{max}} = \frac{1}{2} \sigma_1 = \frac{p \cdot r}{4 \cdot t}.
$$

(7.5-5)

CHAPTER 8 - PRINCIPAL STRESSES UNDER A GIVEN LOADING

8.1 Principal Stresses in a Beam

For beams of rectangular cross section, and within the scope of the material presented in this course, the maximum normal stress occurs at the surface of the beam. Table 1 illustrates this point. In this case, principal stresses are calculated at the two transverse sections of the rectangular cantilever beam supporting a concentrated load shown in Figure 1.

<table>
<thead>
<tr>
<th>y/c</th>
<th>$\frac{\sigma_{\text{min}}}{\sigma_m}$</th>
<th>$\frac{\sigma_{\text{max}}}{\sigma_m}$</th>
<th>$\frac{\sigma_{\text{min}}}{\sigma_m}$</th>
<th>$\frac{\sigma_{\text{max}}}{\sigma_m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0</td>
<td>1.000</td>
<td>0</td>
<td>1.000</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.010</td>
<td>0.810</td>
<td>-0.001</td>
<td>0.801</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.040</td>
<td>0.640</td>
<td>-0.003</td>
<td>0.603</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.090</td>
<td>0.490</td>
<td>-0.007</td>
<td>0.407</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.160</td>
<td>0.360</td>
<td>-0.017</td>
<td>0.217</td>
</tr>
<tr>
<td>0</td>
<td>-0.250</td>
<td>0.250</td>
<td>-0.063</td>
<td>0.063</td>
</tr>
<tr>
<td>-0.2</td>
<td>-0.360</td>
<td>0.160</td>
<td>-0.217</td>
<td>0.017</td>
</tr>
<tr>
<td>-0.4</td>
<td>-0.490</td>
<td>0.090</td>
<td>-0.407</td>
<td>0.007</td>
</tr>
<tr>
<td>-0.6</td>
<td>-0.640</td>
<td>0.040</td>
<td>-0.603</td>
<td>0.003</td>
</tr>
<tr>
<td>-0.8</td>
<td>-0.810</td>
<td>0.010</td>
<td>-0.801</td>
<td>0.001</td>
</tr>
<tr>
<td>-1.0</td>
<td>-1.000</td>
<td>0</td>
<td>-1.000</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. Principal stress distribution at sections located at $r = 2c$ and $r = 8c$ (see Fig. 1).

Curves drawn tangent to the principal stress directions can be established when analyses used to formulate the table are extended to a larger number of sections. These stress trajectories (also called isostatics) are shown for the rectangular cantilever beam in Figure 2.
As opposed to a rectangular section, when the width of the cross section varies in such a way that large shearing stresses occur at point close to the surface, where the normal stress is also large, a value of principal stress larger than that obtained at the free surface may result. One should be particularly aware of this possibility when selecting S-beams or W-beams (“I” beams).

In this case, one must compute the principal stress in the section where the moment is a maximum; at the junction of the web with the flanges. In addition, if the cross section is not symmetrical about its neutral axis, the largest tensile and compressive stresses will not be equal. Moreover, if the shear force varies along the span, the maximum normal stress may not occur in the section where the moment is a maximum.

Figure 1. A rectangular cantilever beam supporting a concentrated load.

Figure 2. Stress trajectories (isostatics) for a rectangular cantilever beam.
8.2 Design of Prismatic Beams

The design of a prismatic beam usually depends on the applied moment, but there may be some situations where the design of the beam is controlled by the maximum shear force. The design procedure includes:

1. Determination of the allowable values of the normal and shear stresses based on a table of material properties. These values may also be computed by dividing the ultimate stress by a safety factor.

2. Drawing shear and moment diagrams to determine the maximum values and the cross sections on which they act.

3. Determination of the minimum allowable section modulus, \[ S = \frac{I}{c} \] using

\[
S_{\text{min}} = \frac{|M|_{\text{max}}}{\sigma_{all}}.
\]

4. Considering available beam sections with \( S > S_{\text{min}} \) and selecting from this group the section with the smallest weight per unit length. This is not necessarily the section with the smallest value of \( S \), since the design may be limited by factors such as the allowable depth of the cross section, the allowable deflection, etc.

5. Checking the resistance to shear of the beam selected.

6. In the case of S-beams or W-beams, checking the value of maximum stress at the junction of the web in the section where the maximum bending moment occurs.

**Example:** See Sample Problem 8.2 on page 521 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.

8.3 Stresses under Combined Loadings

Recall that Chapters 1 and 2 dealt with stresses caused by an axial centric load. The distribution of stresses in a cylindrical member subjected to a twisting couple was studied in Chapter 3. Chapter 4 dealt with stresses caused by bending couples, and, Chapter 6 discussed the stresses produced by transverse loads. This knowledge may be combined to determine the stresses in slender structural members under fairly general loading systems.

Consider the bent member shown in Figure 3. The first step in the analysis of the stresses at section K, for example, is to create an equivalent force couple system at the centroid of the cross section. In this case, the force \( P \) is an axial centric force which produces normal stresses in the section. The couple vectors along \( y \) and \( z \) cause the member to bend and also produce normal
stresses in the section. The normal stresses along x at point K is the sum of the stresses produced by the force and couples (shown below). On the other hand, the twisting couple T and the shearing forces along y and z produce shear stresses in that section. The components of the shearing stress at K may be obtained by adding the corresponding components of the stresses produced at K by each of the forces and couple.

Figure 3. A bent member is subjected to a number of forces.

CHAPTER 9 - DEFLECTION OF BEAMS

9.1 Equation of the Elastic Curve

Recall that a prismatic beam in pure bending bends into a circular arc with curvature

\[
\frac{I}{\rho} = \frac{M(x)}{E I}.
\]  

(9.1-1)

For pure bending, \( M(x) \) is constant over the span. However, when the beam is subjected to a transverse loading, Eqn. (9.1-1) remains valid but both the bending moment and curvature of the neutral surface vary from section to section.

The elastic curve of the beam is the relation between the deflection \( y \) measured at a given point \( Q \) on the axis of the beam and the distance \( x \) of that point from some fixed origin. The elastic curve is illustrated for a beam in Figure 1.

![Elastic Curve Diagram](image)

**Figure 1.** The elastic curve represents the deflection of the beam.

It can be shown from calculus that

\[
\frac{1}{\rho} = \frac{d^2 y}{dx^2}.
\]  

(9.1-2)

and Eqn. (9.1-4) becomes

\[
\frac{d^2 y}{dx^2} = \frac{M(x)}{E I}.
\]  

(9.1-3)

9.1
For a prismatic beam in which E and I are constant, this second order linear differential equation can be integrated to obtain the parametrical representation of the elastic curve as follows

\[ E I \frac{dy}{dx} = E I \theta(x) = \int_0^x M(x) \, dx + C_1 \]  \hspace{1cm} (9.1-4)

where \( \frac{dy}{dx} \) is the slope of the curve and

\[ E I \ y = \int_0^x \left[ \int_0^x M(x) \, dx + C_1 \right] \, dx + C_2 = \int_0^x dx \int_0^x M(x) \, dx + C_1 x + C_2 \]  \hspace{1cm} (9.1-5)

The constants are determined from the boundary conditions imposed on the deflection and slope by considering the beam and the position of its supports. The three types of supports shown in Figure 2 need be considered for complete analysis of statically determinate beams.

**Figure 2.** The three basic supports required to determine the equations of the elastic curve.

Concentrated loads, reactions at supports, or discontinuities in a distributed load make it necessary to divide the beam into several portions, and to represent the moment by a different function \( M(x) \) in each portion of the beam. Additional equations are generated at the interface between different sections, since the slope and deflection must be continuous at these locations (the shear and bending moment may be discontinuous).

Recall that in statically indeterminate problems, the reactions may be obtained by considering the deformations of the structure involved. In these cases, the equilibrium equations may be combined with the expressions for slope and deflection to evaluate both the constants in the parametrical equations and the reactions at points of constraint.

**Example:** See Sample Problem 9.1 on page 563 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.
9.2 Method of Superposition

It is often convenient to compute separately the slope and deflection caused by several concentrated or distributed loads and then to apply the principle of superposition and add the values of slope or deflection. Most structural and mechanical engineering handbooks include tables giving slopes and deflections of beams for various loadings and supports. Table 1 is a typical example.

<table>
<thead>
<tr>
<th>Beam and Loading</th>
<th>Elastic Curve</th>
<th>Maximum Deflection</th>
<th>Slope at End</th>
<th>Equation of Elastic Curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>$\frac{PL^3}{3EI}$</td>
<td>$\frac{PL^2}{2EI}$</td>
<td>$y = \frac{P}{6EI} \left[ x^3 - 3Lx^2 \right]$</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>$\frac{wL^3}{6EI}$</td>
<td>$-\frac{wL^2}{6EI}$</td>
<td>$y = -\frac{w}{24EI} \left( x^4 - 4Lx^3 + 6L^2x^2 \right)$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$\frac{ML^2}{2EI}$</td>
<td>$\frac{ML}{EI}$</td>
<td>$y = \frac{M}{2EI} x^2$</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$\frac{P}{48EI}$</td>
<td>$\frac{PL^2}{16EI}$</td>
<td>For $x &gt; \frac{L}{2}$: $y = \frac{P}{48EI} \left( 4x^3 - 3L^2x \right)$</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>$\frac{PL^3}{36EI}$</td>
<td>$\frac{PL^2}{16EI}$</td>
<td>For $x &lt; \frac{L}{2}$: $y = -\frac{PL}{6EI} \left( x^2 - 6Lx + 2L^2 \right)$</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>$\frac{5wL^3}{384EI}$</td>
<td>$\frac{wL^2}{24EI}$</td>
<td>$y = -\frac{w}{24EI} \left( x^4 - 2Lx^3 + L^2x \right)$</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>$\frac{ML}{9\sqrt{3}EI}$</td>
<td>$\frac{ML}{6EI}$</td>
<td>$y = -\frac{M}{8EI} \left( x^3 - L^2x \right)$</td>
</tr>
</tbody>
</table>

Table 1. Beam deflections and slopes.

9.3
9.3 Application of Superposition to Statically Indeterminate Beams

Statically indeterminate beams can be solved by designating one or more of the reactions as redundant and eliminating or modifying accordingly the corresponding support. The slope or deflection where the support has been modified or eliminated is obtained by computing separately the deformations caused by the given loads and by the redundant reaction, and by superimposing the results obtained.

10.1 Stability of Structures - Euler's Formulas

In discussing the analysis and design of structures in the preceding chapters, we had two primary concerns: (1) the ability of the structure to support a specified load without experiencing excessive stress and (2) the ability of the structure to support a given load without undergoing unacceptable deformation. In this chapter, we shall be concerned with the stability of the structure, i.e., with its ability to support a given load without experiencing a sudden change in its configuration. The discussion will be confined to the analysis and design of vertical prismatic members supporting axial loads. Such structures are called columns.

A column will buckle when it is subjected to a load greater than the critical load denoted by $P_{cr}$. That is, instead of remaining straight, it will suddenly become sharply curved as illustrated in Figure 1.

![Figure 1](image)

Figure 1. A column will buckle when a critical load is reached.

The critical load is given in terms of an effective length by,

$$P_{cr} = \frac{\pi^2 EI}{L^2_e}.$$  \hspace{1cm} (10.1-1)

This expression is known as Euler's formula. The effective length depends upon the constraints imposed on the ends of the column. Table 1 shows the effective length of a column for various end conditions.
10.2 Design of Columns under a Centric Load

It has been assumed in prior analysis of columns that all stresses remained below the elastic limit and that the column was ideally a straight homogeneous prism. Real columns fall short of such an idealization, and in practice the design of columns is based on empirical formulas that reflect the results of numerous laboratory tests. Typical results are shown in Figure 2.

For long columns, with a large effective slenderness ratio, Euler's formula is adequate for design purposes. For intermediate and short columns, where failure occurs essentially as a result of yield, empirical formulas are used to approximate test data. These empirical formulas are specified on the basis of material tests conducted by engineers working in that field.
For example, the American Institute of Steel Construction sets the design standards for structural steel based on the curve shown in Figure 3 and incorporates a safety factor of 1.67. The critical stress in portion AB of the curve is based on an exponential expression [defined below in Eqn. (10.2-2)] while an Euler-based relation [defined in Eqn. (10.2-4)] is used to establish the critical stress in portion BC.

A critical slenderness ratio, $C_c$, is defined as the value of $L_e/r$ at point B. Referring to Figure 3,

\[
C_c = 4.71 \sqrt[2]{\frac{E}{\sigma_y}}.
\]  

(10.2-1)

The procedure for determining the allowable stress and allowable load of a steel column conforming to AISC standards is to first compute the critical slenderness ratio based on the elastic modulus and yield stress by using Eqn. (10.2-1). Then compute the actual slenderness ratio, $L_e/r$, based on the end conditions and the minimum moment of inertia of the column by recalling $I = Ar^2$.

If the slenderness ratio of the column is smaller than $C_c$, a critical stress is computed from the equation,

\[
\text{for } L/r \leq C_c : \quad \sigma_{cr} = \left[ 0.658 \left( \frac{\sigma_e}{\sigma_y} \right) \right] \sigma_y
\]  

(10.2-2)

where

\[
\sigma_e = \frac{\pi^2 E}{(L_e/r)^2}.
\]  

(10.2-3)
If, on the other hand, the slenderness ratio of the column is greater than $C_c$, the critical stress is computed from the equation,

$$ \sigma_{cr} = 0.877 \sigma_e $$  \hspace{1cm} (10.2-4)

In both cases, the allowable stress is found by incorporating a safety factor of 1.67 as follows:

$$ \sigma_{all} = \frac{\sigma_{cr}}{1.67} $$  \hspace{1cm} (10.2-5)

The allowable load is given by,

$$ P_{all} = \sigma_{all} A $$  \hspace{1cm} (10.2-6)

where $A$ is the cross sectional area of the column.

The formulas presented above may be used with either U.S. or SI units to determine the allowable axial stress and allowable load for a given grade of steel and any given allowable value of $L_e/r$. For safety considerations, the steel industry typically places an upper limit of 200 on the slenderness ratio.

Other standards are set for aluminum by the Aluminum Association and for timber by the American Institute of Timber Construction.

**Example:** See Sample Problem 10.02 on page 664 in the 6th edition of Beer, Johnson, DeWolf, and Mazurek.
CHAPTER 11 - ENERGY METHODS

11.1 Strain Energy

Strain energy considerations provide an alternative method of defining the relations that exist between forces and deformations under various loading conditions. In the case of the rod of length L and cross-sectional area A shown in Figure 1, the strain energy is equal to the work done as the load P is slowly applied. As illustrated in Figure 2, it is equal in magnitude to the area under the load-deformation diagram.

**Figure 1.** A rod subjected to an axial load.  
**Figure 2.** Strain energy is equal to the area under the load-deformation diagram.

Hence,

\[
\text{Strain energy} = U = \int_0^{x_1} P \, dx.
\]  

(11.1-1)

If SI units are used, strain energy is expressed in Nm; this unit is called a joule (J). If U.S. customary units are used, work and energy are expressed in ft-lb or in-lb.

11.2 Strain-Energy Density

As noted above, strain energy depends upon geometrical factors (L and A). To eliminate the effect of size and direct attention to the properties of the material, strain energy per unit volume (strain energy density) is considered. Figure 3 illustrates that the strain energy density is equal to the area under the stress strain curve.
Hence, 

\[ \text{Strain-energy density} = u = \int_{0}^{\varepsilon_1} \sigma_x \, d\varepsilon_x \]  \hspace{1cm} (11.2-1) 

The strain energy density is expressed in J/m$^3$ when the SI system is used, and is expressed in in-lb/in$^3$ when U.S. customary units are used.

As illustrated in Figure 4, the strain energy density obtained by setting $\varepsilon_1 = \varepsilon_R$ is the strain at rupture, and is known as the \textit{modulus of toughness}. This property represents the energy per unit volume required to cause the material to rupture. The value of the strain-energy density obtained by setting $\sigma_1 = \sigma_Y$, where $\sigma_Y$ is the yield strength, is called the \textit{modulus of resilience} and represents the energy per unit volume that the material may absorb without yielding. The latter is illustrated in Figure 5.

**Figure 3.** Strain energy density is equal to the area under the stress-strain curve.

**Figure 4.** The modulus of toughness is the strain energy density to rupture.

**Figure 5.** The modulus of resilience is the strain energy density to yield.
11.3 Elastic Strain Energy for Normal Stresses

The elastic strain energy (for elastic deformations) is given by,

\[ U = \int \frac{\sigma_x^2}{2E} dV \].

(11.3-1)

Under axial loading

\[ U = \int_0^L \frac{P^2}{2AE} dx \].

(11.3-2)

For bending

\[ U = \int_0^L \frac{M^2}{2EI} dx \].

(11.3-3)

For shear

\[ U = \int \frac{\tau_{xy}^2}{2G} dV \].

(11.3-4)

For torsion

\[ U = \int_0^L \frac{T^2}{2GJ} dx \].

(11.3-5)

APPENDIX I

by

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APPENDIX I - REVIEW OF STATICS

A1.1 Vectors

Composition refers to the addition of vectors and is performed using the parallelogram law or the triangle rule. Composition of three or more vectors requires successive application of the parallelogram law or the polygon rule.

Resolution, on the other hand, refers to the separation of a vector into its components.

The laws of sines and cosines are often used when composing and/or resolving forces. Referring to Figure 1:

\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}
\]  

(A1.1-1)

and

\[
c^2 = a^2 + b^2 - 2ab \cos \gamma
\]  

(A1.1-2)

Figure 1. The law of sines and cosines can be formulated for any triangle.

Similar relationships can be written for sides a and b.

A1.2 2-D Rectangular Coordinates

One of the most effective methods for working with concurrent coplanar forces is to resolve the forces into components along orthogonal coordinate axes. Referring to Figure 2,

\[
F_x = F \cos \theta_x \quad F_y = F \sin \theta_x
\]

\[
F = \sqrt{F_x^2 + F_y^2} \quad \tan \theta_x = \frac{F_y}{F_x}
\]

(A1.2-1)
When using rectangular coordinates, a scalar component is assumed to be positive when its corresponding vector is in a positive coordinate direction; otherwise, it is negative.

A1.3 3-D Rectangular Coordinates

When adding three or more forces, there is no practical trigonometric solution from the polygon that describes the resultant. The best approach for finding the resultant is to resolve each force into rectangular components. The components are then composed into their resultant.

Figure 2 shows the case in which the line of action of a force is defined by two points.

Figure 3 shows the case in which the line of action of a force is specified by two points along its line of action.

The force may be resolved into rectangular components using a scalar approach based on the force multiplier method. In this case:

\[
\begin{align*}
F &= F_x = F_y = F_z \\
\frac{d}{d_x} &= \frac{d}{d_y} = \frac{d}{d_z}
\end{align*}
\]  \hspace{1cm} (A1.3-1)

where

\[
\begin{align*}
&d_x = x_2 - x_1 \\
&d_y = y_2 - y_1 \\
&d_z = z_2 - z_1 \\
&d = \sqrt{d_x^2 + d_y^2 + d_z^2}
\end{align*}
\]  \hspace{1cm} (A1.3-2)
An alternate approach is based on unit vectors. In this case,

\[ \mathbf{F} = F \hat{\mathbf{r}}_{PQ} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \]  

(A1.3-3)

where

\[ \hat{\mathbf{r}}_{PQ} = \frac{\mathbf{PQ}}{PQ} = \frac{(x_2 - x_1) \mathbf{i} + (y_2 - y_1) \mathbf{j} + (z_2 - z_1) \mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}. \]  

(A1.3-4)

### A1.4 Addition of Concurrent Forces in Space/3-D Equilibrium of Particles

When two or more forces act on the same particle, the resultant can be determined from their components; since,

\[
R_x = \sum F_x \quad R_y = \sum F_y \quad R_z = \sum F_z
\]

\[
R = \sqrt{R_x^2 + R_y^2 + R_z^2}
\]

\[
\cos \theta_x = \frac{R_x}{R} \quad \cos \theta_y = \frac{R_y}{R} \quad \cos \theta_z = \frac{R_z}{R}.
\]

(A1.4-1)

Equilibrium exists when a particle remains at rest or moves with a constant velocity. In this case,

\[
\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0
\]

(A1.4-2)

The method of attack is to resolve all forces into their components, sum the components and then apply the appropriate equations.

### A1.5 Algebraic Manipulation

The scalar or dot product of two vectors \( \mathbf{P} \) and \( \mathbf{Q} \) is defined by

\[ \mathbf{P} \cdot \mathbf{Q} = P \cdot Q \cos \theta \]  

(A1.5-1)

The result of this operation is a scalar. The dot product can be used to find the angle between two vectors, since,

\[ \cos \theta = \frac{\mathbf{P} \cdot \mathbf{Q}}{P \cdot Q} = \frac{\hat{\mathbf{r}}_P \cdot \hat{\mathbf{r}}_Q}{P \cdot Q}. \]  

(A1.5-2)
It is also possible to determine the component of a vector, say $\mathbf{P}$, in any direction in space. The operation involves constructing a unit vector in the desired direction, say $\hat{\lambda}$, and then taking the dot product between $\mathbf{P}$ and $\hat{\lambda}$. Mathematically, denoting the component by $P_{\hat{\lambda}}$,

$$P_{\hat{\lambda}} = \mathbf{P} \cdot \hat{\lambda}. \tag{A1.5-3}$$

In general, two vectors are perpendicular if their dot product is zero. For the unit vectors in a rectangular coordinate system:

$$
\begin{array}{ccc}
1 & \hat{i} \cdot \hat{i} = 1 & \hat{i} \cdot \hat{j} = 0 & \hat{i} \cdot \hat{k} = 0 \\
0 & \hat{j} \cdot \hat{i} = 0 & \hat{j} \cdot \hat{j} = 1 & \hat{j} \cdot \hat{k} = 0 \\
0 & \hat{k} \cdot \hat{i} = 0 & \hat{k} \cdot \hat{j} = 0 & \hat{k} \cdot \hat{k} = 1 \\
\end{array} \tag{A1.5-4}
$$

The vector cross product of $\mathbf{P}$ and $\mathbf{Q}$, $\mathbf{V}$, is defined as:

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} = P Q \sin \theta \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}. \tag{A1.5-5}$$

For the unit vectors in a rectangular system:

$$
\begin{array}{ccc}
0 & \hat{i} \times \hat{i} = 0 & \hat{i} \times \hat{j} = \hat{k} & \hat{i} \times \hat{k} = -\hat{j} \\
-1 & \hat{j} \times \hat{i} = -\hat{k} & \hat{j} \times \hat{j} = 0 & \hat{j} \times \hat{k} = \hat{i} \\
0 & \hat{k} \times \hat{i} = \hat{j} & \hat{k} \times \hat{j} = -\hat{i} & \hat{k} \times \hat{k} = 0 \\
\end{array} \tag{A1.5-6}
$$

The cross product is used when computing moments; $\mathbf{V}$ is perpendicular to the plane formed by $\mathbf{P}$ and $\mathbf{Q}$; $\mathbf{V}$ has magnitude $V = P Q \sin \theta$ which is equal to the area of the parallelogram formed by $\mathbf{P}$ and $\mathbf{Q}$; and, $\mathbf{V}$ forms a right handed triad with the vectors $\mathbf{P}$ and $\mathbf{Q}$.

The scalar triple product is defined as:

$$S \cdot (\mathbf{P} \times \mathbf{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}. \tag{A1.5-7}$$

The scalar triple product is equal to the volume of the parallelopiped formed by the three vectors $S$, $P$ and $Q$. 

A1.4
The scalar triple product may also be used to find the common perpendicular, say \( d \), between \( P \) and \( Q \). This is accomplished by choosing any vector, say \( r \), which joins the lines of actions of the vectors. Then,

\[
d = \frac{r \cdot (P \times Q)}{|P \times Q|}.
\]  

(A1.5-8)

The operation can also be used to determine the moment of a force about a line \( AB \). To obtain \( M_{AB} \), compute a unit vector, \( \lambda_{AB} \), along the line and choose a point on \( AB \), say \( A \). Then,

\[
M_{AB} = \lambda \cdot M_{AB} = \lambda_{AB} \cdot (r_A \times F).
\]  

(A1.5-9)

Finally, the scalar triple product can be used to determine the component of a force \( F \) perpendicular to a plane, say \( DAC \). In this case, two noncollinear and nonparallel vectors are chosen in the plane, say \( DA \) and \( AC \). Then,

\[
F_{\text{perpendicular}} = F \cdot \frac{(DA \times AC)}{|DA \times AC|} = F \cdot \lambda_{\text{perpendicular}}.
\]  

(A1.5-10)

A1.6 Moments and Couples

Referring to Figure 4, the moment of a force \( F \) acting at point \( A \) about point \( O \) may be expressed as either

\[
+\hspace{1em} M = F \cdot d
\]  

(A1.6-1)

or

\[
M_o = r \times F
\]  

(A1.6-2)

where \( d \) is the perpendicular distance from the reference point \( O \) to the line of action of the force \( F \); and, \( r \) is a position vector drawn from \( O \) to any point along the line of action of \( F \).

These equations can also be applied to compute the moment produced by the couple shown in Figure 5.

*Varigon's theorem* states that the moment of a force about any axis is equal to the moments of its components about that axis. Conversely, the moment of several concurrent forces about any axis is equal to the sum of the moment produced by their resultant about that axis.

It should be noted that the maximum moment for a given force, or the least force necessary to produce a given moment, occurs when the force is perpendicular to the moment arm.
A1.7 Force-Couple Systems

The procedure for finding the force in the force-couple system equivalent to the distributed forces and moment shown in Figure 6 is to move all of the forces to $O$ and find the resultant of the resulting concurrent force system. The moment in the force-couple system is determined by adding the moments of all of the forces in the system about the point $O$ to any applied couples.

A1.8 Equilibrium of Rigid Bodies

In general, distributed forces and applied couples cause the rigid body to translate and rotate. For the body to remain at rest or to move with a constant velocity,
These are vector equations with corresponding to six scalar equations; namely,

$$\sum F = 0 \quad \sum M = 0 \quad . \quad (A1.8-1)$$

When dealing with a 2-D force and moment system, the equilibrium equations become:

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_x = \sum M_y = 0 \quad . \quad (A1.8-2)$$

where A is an arbitrary point in space, usually taken in the X-Y plane. The other three equilibrium equations are automatically satisfied. Alternate equations may be applied to determine these unknowns provided that they result in independent equations. One could use, for example,

$$\sum F_x = 0 \quad \sum M_A = 0 \quad \sum M_B = 0 \quad (A1.8-4)$$

or

$$\sum M_A = 0 \quad \sum M_B = 0 \quad \sum M_C = 0 \quad . \quad (A1.8-5)$$

To eliminate unknowns and thus simplify the analytical work, select the moment center so that the line of action of one or more unknowns passes through the point and/or align an axis with one of the unknowns and take the summation of forces in the perpendicular direction.

A1.9 Truss Problems

Figure 7 illustrates a truss which consists of straight members connected at joints.

Figure 7. A truss consists of straight two-force members connected at joints.
No member is continuous through a joint and all applied loads act at the joints. If the weight of the member is to be included, half of its weight is applied to each of the end joints. The force in each member is directed along the member and puts the member in either tension or compression. A tensile member pulls away from its end joints while a compression member pushes toward them. During a truss analysis, it is beneficial to assume that the force in a member always acts away from the end joints. If the magnitude of the force is found to be positive, the member is in tension; otherwise, it is in compression.

The method of joints involves drawing a FBD of the entire truss and then using the 2-D equilibrium equations to solve for the external reactions at the points of constraint. The truss is initially inspected for zero force members; such members are found by inspecting the truss for unloaded joints with three connected members, two of which are collinear. In this case, the force in the third member is zero. The solution continues by starting with a joint which has a maximum of two unknown forces and applying the equilibrium equations for a concurrent system. This procedure is applied to successive joints until all unknowns are determined.

The method of joints is acceptable when the forces in all members are to be determined; however, if the forces in only a limited number of members are required, the method of sections may be the best approach. In some cases, especially when the truss is not simple, the method of sections may offer the only solution to the problem.

The method of attack is to choose a portion of the truss as a free body by passing a section through a maximum of three unknown members; apply overall equilibrium to determine the external reactions only if necessary; identify all zero force members; and, treat the cut members as unknown external forces and apply the conventional 2-D equilibrium equations.

An important point to remember is that if three members are cut, taking moments about the point of intersection of the lines of action of two of the members allows the force in the third member to be determined in an independent equation. Also note that the method of sections can be applied to find the force in three members and then the method of joints can be applied to find the forces in other members of interest.

A1.10 Frames and Machine Problems

The method of members is applied to analyze structures which contain at least one member which is multiforce. That is, the member is acted upon by three or more forces, one of which is acting at a point other than a joint. The associated structures are classified either as frames that are usually constrained and used to support loads; or, machines that are used to transmit or modify loads and may or may not be stationary.

The method of attack for the method of members is to first consider the structure as a whole and use external equilibrium to determine the reactions at the points of constraint; then, consider each portion of the structure as a separate free body, denoting and labeling forces consistent with Newton's third law. During this step, it is beneficial to look for two force members. When the
forces and reactions are properly labeled and the various parts of the structure combined, only
the external forces and the reactions remain.

A1.11 Friction

**Figure 8** shows a block resting on a rough surface. The forces acting on the block are its weight \( W \) and the reaction at the surface; \( N \) is normal to the surface as a consequence of the free body
diagram.

**Figure 9**, on the other hand, shows the case in which a small force, \( P \), acts horizontally on the
block. If \( P = 0; \) \( F = 0 \) and \( W = N \), as in the previous case. If \( P \) is small enough so the block does
not move, \( F = P \); and, \( F \) is called the static friction force. When the block is about to move \( P = F_m \) (\( F \) reaches its maximum at this point). When the block moves, \( P \) drops from \( F_m \) to \( F_k \); \( F_k < F_m \), because there is less interpenetration between the irregularities of the surfaces in contact
when the surfaces are moving with respect to each other. From then on, \( F_k \), the kinetic friction
force, remains constant while the velocity increases. These results are summarized in **Figure 10**.

![Figure 8](image1)
**Figure 8.** A block rests on a horizontal surface.

![Figure 9](image2)
**Figure 9.** A progressively larger horizontal force is applied.

![Figure 10](image3)
**Figure 10.** A plot of \( P \) versus \( F \).

From experimental tests:

\[
F_m = \mu_s N, \quad F_k = \mu_k N
\]  \hspace{1cm} (A1.11-1)

where \( \mu_s \) and \( \mu_k \) depend on the nature of the surface, as opposed to the area; and,

\[
\mu_k = 0.75 \mu_s
\]  \hspace{1cm} (A1.11-2)

Previously, \( P \) was considered horizontal, however, four different situations may occur when a
force is applied to the block as shown in **Figures 11 through 14**.
As shown in Figures 15 through 18, it is sometimes convenient to replace \( F \) and \( N \) by their resultant \( R \). In Figures 16 through 18, \( \varphi_s \) and \( \varphi_k \) are the angles of static and kinetic friction, respectively; and,

\[
\tan \varphi_s = \mu_s \quad \text{and} \quad \tan \varphi_k = \mu_k \quad .
\]

The angle of response, \( \varphi_s \), is the angle of inclination corresponding to the condition of impending motion of a block resting on an inclined surface. The angle is obtained experimentally by placing a block on a plane which is progressively inclined. The above equation is then used to compute the coefficient of static friction.

The solution to friction problems relies on the equilibrium equations. The sense of the friction force acting on a surface is shown on the FBD opposite to the direction of motion. It should be noted that the minimum force required to start a block resting on a rough horizontal surface occurs when the force is perpendicular to the reaction at the surface.

To make things simple: when \( N \) can be computed directly in terms of known forces, use \( F \) and \( N \); otherwise use \( R \) at the appropriate angle; when three forces act on a body, the law of sines,
cosines, and the force triangle approach may be of benefit; if more than three forces are involved, summing forces perpendicular to the line of action of one unknown will eliminate that unknown from the equation; always consider eliminating reactions and unknown forces by taking moments about a point on their line of action.

Figure 15. No friction ($P_x = 0$).

Figure 16. No motion ($P_x > 0$).

Figure 17. Motion impending ($P_x = F_m$).

Figure 18. Motion ($P_x > F_m$).

Figure 19 illustrates the condition where a flat belt passes over a fixed cylindrical surface.

Figure 19. Friction occurs when a belt passes over a fixed surface.
The tensions in the two parts of the belt when it is about to slide to, say, the right are given by,

\[
\ln \frac{T_2}{T_1} = \mu_s \beta \quad \text{or} \quad \frac{T_2}{T_1} = e^{\mu_s \beta}
\]  \hspace{1cm} (A1.11-3)

where \( \beta \) is the angle of contact (expressed in radians) and \( \mu_s \) is the coefficient of static friction. \( T_2 \) is algebraically larger than \( T_1 \); consequently, \( T_2 \) represents the tension in that part of the belt or rope which pulls, while \( T_1 \) is the tension in the portion which resists. If the belt or rope is slipping, similar relations hold true with \( \mu_s \) replaced with \( \mu_k \).

**A1.12 Center of Gravity and Centroids**

For the purposes of computing the reactions at the points of constraint, the distributed weight of a body can be considered to be a concentrated force acting through a single point called the center of gravity. For the flat plate shown in **Figure 20**:

\[
\bar{x} = \frac{\sum_{i=1}^{n} x_i w_i}{W} = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i} \quad \bar{y} = \frac{\sum_{i=1}^{n} y_i w_i}{W} = \frac{\sum_{i=1}^{n} y_i w_i}{\sum_{i=1}^{n} w_i}
\]  \hspace{1cm} (A1.12-1)

**Figure 20.** Finding the centroid of a flat plate located in the XY plane.

For a homogeneous plate of constant thickness, the center of gravity is called the centroid. In this case:
As the number of plates increases to infinity, the plates have differential areas and the finite sum becomes an integral. In this case,

\[
- \bar{x} = \frac{\int x_{el} \, dA}{A} = \frac{\int x_{el} \, dA}{\int dA} = \frac{\int y_{el} \, dA}{A} = \frac{\int y_{el} \, dA}{\int dA} = \bar{y} .
\]  

The expressions \(x_{el}dA\) and \(y_{el}dA\) are called the first moment of the area about the Y and X axis, respectively. If the centroid is located on a coordinate axis, the first moment of the area with respect to that axis is zero. The converse also holds true.

For a homogeneous wire of constant cross section:

\[
- \bar{x} = \frac{\sum_{i=1}^{n} x_{i} L_{i}}{L} = \frac{\sum_{i=1}^{n} x_{i} L_{i}}{\sum_{i=1}^{n} L_{i}} = \frac{\sum_{i=1}^{n} y_{i} L_{i}}{L} = \frac{\sum_{i=1}^{n} y_{i} L_{i}}{\sum_{i=1}^{n} L_{i}} .
\]  

Following a similar argument to that taken for the plate:

\[
- \bar{x} = \frac{\int x_{el} \, dL}{L} = \frac{\int x_{el} \, dL}{\int dL} = \frac{\int y_{el} \, dL}{L} = \frac{\int y_{el} \, dL}{\int dL} .
\]  

The above expressions specify the coordinates of the centroid of a homogeneous wire of constant cross section which coincide with its center of gravity.

Similar arguments hold true for 3-D to those derived previously in 2-D. The center of gravity of a 3-D body lies at:

\[
- \bar{x} = \frac{\sum_{i=1}^{n} x_{i} W_{i}}{W} = \frac{\sum_{i=1}^{n} x_{i} W_{i}}{\sum_{i=1}^{n} W_{i}} = \frac{\sum_{i=1}^{n} y_{i} W_{i}}{W} = \frac{\sum_{i=1}^{n} y_{i} W_{i}}{\sum_{i=1}^{n} W_{i}} = \frac{\sum_{i=1}^{n} z_{i} W_{i}}{W} = \frac{\sum_{i=1}^{n} z_{i} W_{i}}{\sum_{i=1}^{n} W_{i}} .
\]
Or,

\[
\begin{align*}
-\bar{x} &= \frac{\int x_{el} \, dw}{W} = \int x_{el} \, dw \int dw, \\
-\bar{y} &= \frac{\int y_{el} \, dw}{W} = \int y_{el} \, dw \int dw, \\
-\bar{z} &= \frac{\int z_{el} \, dw}{W} = \int z_{el} \, dw \int dw.
\end{align*}
\]  \hspace{1cm} (A1.12-7)

When the body is homogeneous:

\[
\begin{align*}
\bar{x} &= \frac{\sum_{i=1}^{n} x_i V_i}{V} = \frac{\sum_{i=1}^{n} x_i V_i}{\sum_{i=1}^{n} V_i}, \\
\bar{y} &= \frac{\sum_{i=1}^{n} y_i V_i}{V} = \frac{\sum_{i=1}^{n} y_i V_i}{\sum_{i=1}^{n} V_i}, \\
\bar{z} &= \frac{\sum_{i=1}^{n} z_i V_i}{V} = \frac{\sum_{i=1}^{n} z_i V_i}{\sum_{i=1}^{n} V_i}.
\end{align*}
\]  \hspace{1cm} (A1.12-8)

Or,

\[
\begin{align*}
-\bar{x} &= \frac{\int x_{el} \, dV}{V} = \int x_{el} \, dV \int dV, \\
-\bar{y} &= \frac{\int y_{el} \, dV}{V} = \int y_{el} \, dV \int dV, \\
-\bar{z} &= \frac{\int z_{el} \, dV}{V} = \int z_{el} \, dV \int dV.
\end{align*}
\]  \hspace{1cm} (A1.12-9)

The center of gravity or centroid is determined using either the method of integration or the method of composites. When the method of composites is used, a rectangular or polar coordinate axes system is established. The coordinates of the centroid of each individual portion of the area are assigned as positive or negative depending upon their position with respect to the coordinate axes system. Areas and/or lengths are considered as positive or negative depending upon whether the shape must be added or subtracted, respectively, to form the overall shape desired.

In general, it is necessary to perform a double or a triple integration to evaluate the integrals in question. In two dimensions, the differential area in rectangular coordinates is (dxdy); whereas, in polar coordinates, it is (rdrdθ). In many cases, however, a single integration can be used. In the case of an area, for example, the method of attack is to choose a rectangular strip, or appropriate sector, as an element. For a strip, the coordinates of the centroid \((x_{el},y_{el})\) and the area \((dA)\) are expressed in terms of the coordinates and their differentials. In three dimensions, a similar approach can be applied if the body is one of revolution. In this case, a thin disk is taken as a differential element of volume.

**A1.13 Moments of Inertia**

Inertia has significance when used in conjunction with other quantities (equations of motion in dynamics, flexure formula for beam stress in mechanics of materials, etc.). Basically, the term inertia is used to describe the tendency of matter to show resistance to change.

Referring to the X and Y axes shown in Figure 21, the area moments of inertia are:
\[ I_x = \int y^2 \, dA \quad I_y = \int x^2 \, dA \] \hspace{1cm} \text{(A1.13-1)}

**Figure 21.** The moment of inertia is defined with respect to a rectangular coordinate system.

The units associated with the area moment of inertia are in^4, cm^4, etc. Since \( \rho^2 > 0 \), the moment of inertia is always positive.

The polar moment of inertia is used to characterize torsional behavior in cylindrical shafts, the rotation of slabs, etc. It is computed for an axis passing through point O perpendicular to the x,y plane (i.e., around a z axis). By definition,

\[ J_O = \int r^2 \, dA = I_x + I_y \] \hspace{1cm} \text{(A1.13-2)}

where \( r \) is the distance from the element of area to the point O.

Imagine that the area shown in **Figure 21** is squeezed either into a strip oriented parallel to one of the coordinate axes as shown in **Figures 22 and 23**, or, into a thin ring centered at O as shown in **Figure 24**.

**Figure 22.** The area is assumed to be a strip parallel to the X axis.

**Figure 23.** The area is assumed to be a strip parallel to the Y axis.

**Figure 24.** The area is assumed to be a thin ring centered at the origin.

A1.15
For the area to have the same $I_x$, $I_y$ and $J_o$, the respective strips must lie at distances $k_x$, $k_y$ and $k_o$ respectively such that,

$$I_x = k_x^2 A \quad I_y = k_y^2 A \quad J_o = k_o^2 A .$$  \hspace{1cm} (A1.13-3)

The distances $k_x$, $k_y$ and $k_o$ are called the radii of gyration. Note that $k_x$ and $k_y$ are measured perpendicular to the $x$ and $y$ axes, respectively, not along them.

As illustrated by Figure 25, it is often convenient to transfer the moment of inertia between a centroidal axis and an arbitrary parallel axis. This transformation can be accomplished using a simple formula known as the parallel axis theorem:

$$I = ar{I} + A d^2$$  \hspace{1cm} (A1.13-4)

where $I$ is the area moment of inertia about the arbitrary axis, $I$ is the moment of inertia about a parallel axis passing through the centroid of the area, and $d$ is the perpendicular distance between the two parallel axes.

![Figure 25](image-url)

**Figure 25.** The moment of inertia is transferred between two parallel axes.

Utilizing the expressions for the radii of gyration, the parallel axis theorem may also be expressed as follows:

$$k^2 = ar{k}^2 + d^2$$  \hspace{1cm} (A1.13-5)

This approach can also be taken to transfer the polar moment of inertia and the polar radius of gyration, respectively, as follows:

$$J_o = ar{J}_o + A d^2 \quad k_o^2 = ar{k}_o^2 + d^2$$  \hspace{1cm} (A1.13-6)

The most simple method for direct integration of the moment of inertia is to choose a strip parallel to the axis under consideration (for $I_x$ choose the strip parallel to the $x$ axis; for $I_y$ choose...
the strip parallel to the y axis), so that the perpendicular distance to each point on the element is the same. Then express the area of the strip, \(dA\), and either \(x_{el}\) or \(y_{el}\) as needed in terms of the coordinates of the problem and integrate.

**Note:** The moment of inertia can be found about one axis by integration, after which it can be transferred to a parallel axis using the parallel axis theorem. It is necessary, however, that one of the axes be centroidal.

If the area under consideration is made up of several parts, its moment of inertia about an axis is found by adding the moments of inertia of the composite areas about that axis. In general, it is necessary to transfer each moment of inertia to the appropriate axis by the parallel axis theorem before adding.

**Note:** The radius of gyration is *not* the sum of the component areas' radius of gyration. It is first necessary to perform the superposition of the moments of inertia and then use the definition the \(k^2 = \frac{I_{\text{total}}}{A_{\text{total}}}\).

The mass moment of inertia is given by,

\[
I = \int r^2 \, dm \quad I = k^2 \, m
\]  
(A1.13-7)

In this case, the parallel axis theorem takes the form,

\[
I = \overline{I} + m \, d^2
\]  
(A1.13-8)

### A1.14 Shear and Moment Diagrams

A *beam* is a structural member that is designed to support loads at various points along its span. In most cases, loads are perpendicular to the beam. A portion of mechanics of materials deals with selecting a cross section to resist these shearing forces and bending moments. This often requires drawing shear and moment diagrams for the beam.

The method of attack is to determine the reactions at points of constraint, then cut the beam at various sections along the span, indicating on which portion of the beam internal forces are acting. A convention must be established; the shear \(V\) and the bending moment \(M\) at a given point of a beam are assumed to be positive when the internal forces and couples acting on each element of the beam are directed as shown in **Figure 26**.

The following relationships are often helpful:

\[
\frac{d \, V}{d \, x} = -w \quad \frac{d \, M}{d \, x} = V
\]  
(A1.14-1)
where $w$ is the distributed load per unit length.

**Figure 26.** The sign convention used for constructing shear and bending moment diagrams.
\[ \sigma_{\text{max,min}} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left( \frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \tau_{xy}^2} \quad \text{and} \quad \tau'_{xy} = 0 \quad \text{with} \quad \tan 2 \theta_p = \frac{2 \tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \]

\[ \sigma_{\text{ave}} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \quad R = \sqrt{\left( \frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \tau_{xy}^2} \quad \tau_{\text{max}} = \frac{1}{2} \left| \sigma_{\text{max}} - \sigma_{\text{min}} \right| \]

**APPENDIX II**

by

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APPENDIX II – BASIC FORMULAS

Axial Loading:

\[ \sigma = \frac{P}{A} \quad \tau_{\text{ave}} = \frac{P}{A} \quad \sigma_p = \frac{P}{A} = \frac{P}{t \, d} \]

Factor of safety \( F.S. = \frac{\text{ultimate load}}{\text{allowable load}} \)

\[ \varepsilon = \frac{\delta}{L} \quad \delta = \frac{PL}{AE} \quad v = -\begin{bmatrix} \text{lateral strain} \\ \text{axial strain} \end{bmatrix} \]

Thermal Deformation:

\[ \delta_T = \alpha (\Delta T) \, L \quad \varepsilon_T = \frac{\delta_T}{L} = \alpha \Delta T \]

Generalized Hooke's Laws:

\[ \varepsilon_x = +\frac{\sigma_x}{E} - \frac{v \sigma_y}{E} - \frac{v \sigma_z}{E} \quad \varepsilon_y = -\frac{v \sigma_x}{E} + \frac{\sigma_y}{E} - \frac{v \sigma_z}{E} \quad \varepsilon_z = -\frac{v \sigma_x}{E} - \frac{v \sigma_y}{E} + \frac{\sigma_z}{E} \]

\[ \gamma_{xy} = \frac{\tau_{xy}}{G} \quad \gamma_{yz} = \frac{\tau_{yz}}{G} \quad \gamma_{zx} = \frac{\tau_{zx}}{G} \quad \text{where} \quad G = \frac{E}{2(1 + v)} \]

Stress Concentration Factor (Axial Loading):

\[ K = \frac{\sigma_{\text{max}}}{\sigma_{\text{ave}}} \]

Torsion of Circular Shafts:

\[ \gamma = \frac{\rho \, \phi}{L} \quad \tau = G \gamma = \frac{T \, \rho}{J} \quad \phi = \frac{T \, L}{J \, G} \]

solid shaft: \( J = \frac{1}{2} \pi \, c^4 \)    hollow shaft: \( J = \frac{1}{2} \pi \, c^2 - \frac{1}{2} \pi \, c_1^4 = \frac{1}{2} \pi \left( c_2^4 - c_1^4 \right) \)

A2.1
Transmission Shafts:

\[ P = 2\pi \, f \, T \quad 1 \text{rpm} = \frac{1}{60} \, \text{s}^{-1} = \frac{1}{60} \, \text{Hz} \quad 1 \text{hp} = 550 \, \text{ft} \cdot \text{lb} / \text{s} = 6600 \, \text{in} \cdot \text{lb} / \text{s} \]

Stress Concentration Factor (Stepped Shaft in Torsion):

\[ \tau_{\text{max}} = K \frac{T \, c}{J} \]

Pure Bending (Homogeneous Beams):

\[ \frac{I}{\rho} = \frac{M}{E \, l} \quad \text{anticlastic curvature} = \frac{1}{\rho} = \frac{v}{\rho} \]

\[ \varepsilon_x = -\frac{y}{\rho} \quad \sigma_x = -\frac{M \, y}{I} \quad \sigma_m = \frac{M}{S} \quad S = \frac{I}{c} \]

Pure Bending (Composite Beams):

\[ n = \frac{E_2}{E_1} \quad \sigma_x = -\frac{n \, M \, y}{I} \quad \frac{1}{\rho} = \frac{M}{E_1 \, l} \]

Stress Concentration Factor (Notched Beam in Bending):

\[ \sigma_m = K \frac{M \, c}{I} \]

Eccentric Axial Loading:

\[ \sigma_x = \frac{P}{A} - \frac{M \, z}{I_z} + \frac{M \, y}{I_y} \quad \tan \phi = \frac{I_z}{I_y} \tan \theta \]

Shear and Moment Considerations:

\[ \sigma_m = \frac{|M|_{\text{max}} \, c}{I} \quad \tau_m = \frac{|V|_{\text{max}} \, Q}{I \, t} \quad \frac{d \, V}{d \, x} = -w \quad \frac{dM}{dx} = V \]

Transverse Loading:

\[ q = \frac{V \, Q}{I} \quad \tau_{\text{ave}} = \frac{V \, Q}{I \, t} \]

A2.2
Transformation Equations:

\[
\sigma_{xx}' = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]

\[
\sigma_{yy}' = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta
\]

\[
\tau_{xy}' = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]

\[
\sigma_{\text{max,min}} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}
\]

\[
\tau_{\text{max}} = \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}
\]

Mohr's Circle:

\[
\sigma_c = \frac{\sigma_{xx} + \sigma_{yy}}{2} \quad \tau_c = 0 \quad R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \quad \tau_{\text{max}} = \frac{1}{2} \left| \sigma_{\text{max}} - \sigma_{\text{min}} \right|
\]

Pressure Vessels:

Cylindrical: \( \sigma_1 = \frac{p r}{t} \quad \sigma_2 = \frac{p r}{2 t} \)

Spherical: \( \sigma_1 = \sigma_2 = \frac{p r}{2 t} \)

Design of Prismatic Beams:

\[
S_{\text{min}} = \frac{|M|_{\text{max}}}{\sigma_{\text{all}}}
\]

Deflection of Beams:

\[
\frac{d^2y}{dx^2} = \frac{M(x)}{EI}
\]

Buckling of Columns:

\[
P_{cr} = \frac{\pi^2 EI}{L_e^2} \quad \sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \quad I = A \ r^2
\]
Design of Steel Columns:

\[ I = A \frac{r^2}{2} \quad C_c = 4.71 \frac{E}{\sigma_y} \]

for \( L_e/r \leq C_c \):
\[
\sigma_{cr} = \left[ 0.658 \left( \frac{\sigma_y}{\sigma_e} \right) \right] \sigma_y
\]
\[
\sigma_{all} = \frac{\sigma_{cr}}{1.67}
\]

for \( L_e/r \geq C_c \):
\[
\sigma_{cr} = 0.877 \sigma_e
\]
\[
\sigma_{all} = \frac{\sigma_{cr}}{1.67}
\]

where
\[
\sigma_e = \frac{\pi^2 E}{(L_e/r)^2}
\]

\[ P_{all} = \sigma_{all} A \]

Conversions:

\[ g = 32.2 \text{ ft/s}^2 = 9.81 \text{ m/s}^2; 1 \text{ slug} = 14.59 \text{ kg}; 1 \text{ ft} = .3048 \text{ m}; 1 \text{ lb} = 4.45 \text{ N}; 1 \text{ psi} = 6.895 \text{ kPa} \]
MAE/CE 370

- SYNOPSIS OF MECHANICS OF MATERIALS -

APPENDIX III

by

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APPENDIX III – SYNOPSIS OF MECHANICS OF MATERIALS

A3.1 Synopsis of Mechanics of Materials

When loads are applied to a deformable body they produce stresses. The stresses represent the force intensity and are computed by dividing the force by the area over which it acts. A normal stress is produced when the force is perpendicular to the surface under consideration. A tensile stress results when the force is directed along the outer normal to the surface; a compressive stress results when the force is directed toward the element. Shear stress results when the force is tangent to the surface.

The stresses produce changes in shape (deformations) characterized by a quantity called strain. Normal stresses produce normal strains defined as the change in length of a line segment divided by the original length of the segment. Shear stresses produce shear strains defined as the change in angle between two line segments that were originally perpendicular to one another.

There are three basic loading conditions that can be superimposed to produce a general state of deformation. Axial loading produces either tensile or compressive stresses assumed to be uniformly distributed across a plane cut normal to the longitudinal axis of the member. Combinations of normal and shear stresses occur on oblique planes. The stresses are computed by dividing the load (P) by the area (A) considered; thus,

\[
\sigma = \frac{P}{A} \quad \tau = \frac{P}{A}
\]

Bending produces a uniaxial stress condition in which normal stresses occur parallel to the longitudinal axis of the member. For a prismatic member possessing a plane of symmetry, subjected at its ends to equal and opposite couples acting in a plane of symmetry, the stress distribution is linear through the thickness; compressive stresses occur on one side of the neutral axis and tensile stress occur on the other side. The stress is computed using the flexure formula

\[
\sigma = \frac{Mc}{I}
\]

where M is the moment, c is the distance measured from the neutral axis to the point under consideration, and I is the centroidal moment of inertia measured around the axis about which the moment is applied.

Torsion produces shear stresses. In a prismatic member of circular cross section subjected to couples (torques) of magnitude T, the shear stress acts in the direction of the applied torque. It is zero at the center of the shaft and maximizes at the outer surface. The stress is computed using the elastic torsion formula

\[
\sigma = \frac{My}{I}
\]
where $\rho$ is the distance measured from the center of the shaft to the point under consideration and $J$ is the polar moment of inertia.

Shear stresses also result in prismatic members subjected to transverse loads. In this case, the stress is given by

$$\tau_m = \frac{|V|_{\text{max}} Q}{I t}$$

where $V$ is the shear force, $Q$ is the first moment of the area measured about the neutral axis of the portion of the cross section located either above or below the point under consideration, $I$ is the centroidal moment of inertia of the entire cross section, and $t$ is the width of a cut made through the point perpendicular to the applied load.

The stress is related to the strain through constitutive equations (Hooke’s Laws) that depend upon material properties. In general, these are very complex relations especially when dealing with composite materials; however, only two independent material constants [typically Young’s modulus (E) and Poisson’s ratio ($\nu$)] are required when dealing with a linearly elastic (stress-strain curve is linear), homogeneous (mechanical properties are independent of the point considered), and isotropic (material properties are independent of direction at a given point) material. Other constants, such as the shear modulus [$G = E/2(1 + \nu)$], are sometimes used in these equations. Material constants are evaluated under simple loading conditions (tension test, bending test, torsion tests) on specimens having simple geometry (typically rods, bars, and beams).

It is often necessary to find the weakest link in a structure and this can be done by studying the stress distribution at a critical location. This is not as easy as it may sound, since stress is a tensor and must be studied using transformation equations. That is, the stress distribution changes from point to point in a loaded member. However, there is usually a critical point or region in the structure where the stress is the highest. The stress distribution at this critical location depends upon the plane considered. There is a set of planes on which the normal stress becomes maximum; shear stress is zero on these planes. The planes of maximum shear typically have a non-zero normal stress component. Analytical equations (transformation equations) or graphical techniques (Mohr’s circle) can be employed to pinpoint the maximum stresses and the planes on which they act. These quantities are used in conjunction with failure criteria to assess structural integrity.

In some cases, a member may fail due to its unstable geometry as opposed to the imposed stress state. Column buckling is one of the areas where such considerations come into play. Design codes have been developed to prevent these cataclysmic failures.
APPENDIX IV – SOLVING PROBLEMS AND SUGGESTED READING

A4.1 Introduction

Many engineering students feel that some mechanics courses, such as statics and dynamics, are the most difficult courses that they have had. In general, it is not because the material is so difficult to understand but because these courses require a systematic approach to problem solving. The objective of the section is to help develop a comprehensive and flexible approach which may be applied to solve every new engineering problem.

A4.2 McMaster Strategy

Several years ago the McMaster University engineering faculty were concerned that their students were not adequately prepared to deal with problems that they would eventually face in industry. In an effort to remedy this, the faculty conducted a study into problem solving techniques. One of the things that came out of their study was the following six step plan:

1. Always think that you want to, and that you can, solve the problem at hand. Motivate yourself and minimize distress. Your goals are to relax and build self confidence.

2. Define the problem as stated. Understand the words, identify the objectives and draw diagrams where applicable. Identify the system along with stated input, output, knowns, unknowns, stated and inferred constraints and criteria. Your goal is to fully understand the problem before attempting a solution.

3. Define and explore the issues. Draw from past experiences. Recall theory and fundamentals that seem pertinent. Hypothesize, visualize, idealize, generalize and simplify. Your goal is to collect all of your resources to attack the problem at hand.

4. Plan the course of action. Select tactics, assemble resources, develop tasks, sub-tasks, etc. Your goal is to organize the work into manageable tasks and establish an overall game plan.

5. Solve the problem.

6. Review the solution. Check for reasonableness and errors. Make sure that the criteria have been satisfied. Your goal is to assess performance and identify problems in the overall approach to prevent them from recurring.
A4.3 How to Win Friends and Influence People

“How to Win Friends and Influence People”
by
Dale Carnegie
Simon and Schuster, 1230 Avenue of the Americas, New York, New York 10020
Copyright, 1936.

A4.4 About Dale Carnegie

CARNEGIE, DALE (1888-1955), was a pioneer in public speaking and personality development. He became famous by showing others how to become successful. His book “How to Win Friends and Influence People” (1936) has sold more than 15 million copies and has been translated into many languages.

His books became popular because of his illustrative stories and simple, well-phrased rules. Two of his most famous maxims are, "Believe that you will succeed, and you will," and "Learn to love, respect and enjoy other people."

A4.5 About Carnegie’s Book

The sole purpose of this book is to help you solve the biggest problem you face: the problem of getting along with and influencing people in your everyday, business and social contacts. This book has sold more than fifteen million copies--one of the greatest records in history for a non-fiction book. Its title has become a phrase in the English language. This book can easily be worth its weight in gold to you.

A4.6 Summary of the Manuscript

Table of Contents

1. Fundamental Techniques in Handling People
2. Six Ways to Make People Like You
3. How to Win People to Your Way of Thinking
4. Be a Leader: How to Change People Without Giving Offense or Arousing Resentment

Part One

Fundamental Techniques in Handling People:

1. Don’t criticize, condemn or complain.
2. Give honest and sincere appreciation.
3. Arouse in the other person an eager want.

**Part Two**

Six ways to make people like you:

1. Become genuinely interested in other people.
2. Smile.
3. Remember that a person's name is to that person the sweetest and most important sound in any language.
4. Be a good listener. Encourage others to talk about themselves.
5. Talk in terms of the other person's interests.
6. Make the other person feel important - and do it sincerely.

**Part Three**

Win people to your way of thinking:

1. The only way to get the best of an argument is to avoid it.
2. Show respect for the other person's opinions. Never say, "You're wrong."
3. If you are wrong, admit it quickly and emphatically.
5. Get the other person saying "yes, yes" immediately.
6. Let the other person do a great deal of the talking.
7. Let the other person feel that the idea is his or hers.
8. Try honestly to see things from the other person's point of view.
9. Be sympathetic with the other person's ideas and desires.
10. Appeal to the nobler motives.
11. Dramatize your ideas.
12. Throw down a challenge.

**Part Four**

Be a Leader: How to Change People without Giving Offense or Arousing Resentment:

A leader's job often includes changing your people's attitudes and behavior. Some suggestions to accomplish this:

1. Begin with praise and honest appreciation.
2. Call attention to people's mistakes indirectly.
3. Talk about your own mistakes before criticizing the other person.
4. Ask questions instead of giving direct orders.
5. Let the other person save face.
6. Praise the slightest improvement and praise every improvement. Be "hearty in your approbation and lavish in your praise."

A4.3
7. Give the other person a fine reputation to live up to.
8. Use encouragement. Make the fault seem easy to correct.
9. Make the other person happy about doing the thing you suggest.
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- CLASSIFICATION OF I-BEAMS -

APPENDIX V

by

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Steel S Section I-Beams

The I-Beams are identified by:
**S DEPTH (inches) × WEIGHT PER UNIT LENGTH (pound force per foot)**
For Example: **S18 × 54.7** is an I-Beam with a Depth of 18 inches and having a Nominal Weight per Foot of 54.7 lbf/ft.

Steel Wide Flange I-Beams

The I-Beams are identified by:
**W DEPTH (inches) × WEIGHT PER UNIT LENGTH (pound force per foot)**
For Example: **W27 × 161** is an I-Beam with a Depth of 27 inches and having a Nominal Weight per Foot of 161 lbf/ft.