HOLOGRAPHIC DISPLACEMENT ANALYSIS USING
WAVE FRONT MODULATION TECHNIQUES

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ABSTRACT

The wave fronts from two holographic displacement patterns are modulated by equal but opposite plate rotations and are superimposed to form a moiré pattern. Displacement information contained in the Fraunhofer diffraction pattern of the two superimposed gratings is analysed, and spatial filtering techniques are used to isolate full-field projections of displacement along and normal to the line of sight. Experimental results taken from a pipe subjected to pure torsion show that the method is not only accurate but that the technique is applicable to curved surfaces.

NOMENCLATURE

d displacement vector
d_{i} inner diameter of pipe
d_{o} outer diameter of pipe
\hat{e}_{i} unit vector in direction of propagation
i, j, k unit vectors
n_{i} fringe order number
r, \theta, z polar coordinates
C centre of rotation
D distance between model and photographic plate
E elastic modulus
L length of pipe
M_T applied torque
P object point
U, V, W, U_r, U_\theta, U_z scalar components of displacement
Holographic interferometry can be used to generate interference fringes on a laser illuminated surface. Points on a given fringe experience equal changes in optical path length between successive holographic recordings. The corresponding phase change is usually due to actual surface deformation; however, additional phase changes can be artificially produced by modulating either the object or reference wave front between exposures. This presents no real problem from an analysis standpoint since the latter can be looked upon as equivalent model deformation.

This paper employs wave front modulation to simultaneously record full-field projections of displacement. This is accomplished by incorporating phase changes caused by equal but opposite plate rotations into two holographic displacement patterns taken from a model illuminated with two object beams. These patterns are simultaneously reconstructed and the resultant superposition is optically filtered to yield two full-field moiré patterns for displacement components along and normal to the line of sight.

Experimental results taken from a cylindrical pipe subjected to pure torsion show that the method is not only accurate but that the technique is applicable to curved surfaces.

ANALYSIS

As shown in Fig. 1, a model is illuminated with two beams contained in the X-Y plane. The two beams, characterised by unit vectors \( \hat{e}_1 \) and \( \hat{e}_2 \), make equal angles \( \alpha \) with respect to the viewing direction given by unit vector \( \hat{e}_3 \). A photographic plate, situated along the observation direction, is used to capture these object wave fronts with a reference beam at angle \( \theta_R \) with respect to the plate normal. The plate is mounted so that rotation can be carried out around
an axis parallel to the plane of the plate which passes through a rotation centre whose position can be varied along \( \hat{e}_3 \).

When the plate remains stationary and points on the object surface are displaced through a vector displacement \( \mathbf{d} \) between photographic recordings, two holograms are generated which display interference fringes given by,\(^1\)

\[
(\hat{e}_1 - \hat{e}_3) \cdot \mathbf{d} = n_1 \lambda
\]  
(1)

and

\[
(\hat{e}_2 - \hat{e}_3) \cdot \mathbf{d} = n_2 \lambda
\]  
(2)

where \( n_1 \) and \( n_2 \) are fringe order numbers and \( \lambda \) is the wavelength of the laser used during the recordings. Each unit vector can be written in terms of the cartesian reference system as,

\[
\hat{e}_1 = \sin \alpha \mathbf{i} - \cos \alpha \mathbf{j}
\]  
(3)

\[
\hat{e}_2 = -\sin \alpha \mathbf{i} - \cos \alpha \mathbf{j}
\]  
(4)

\[
\hat{e}_3 = \mathbf{j}
\]  
(5)

where \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are unit vectors along \( X, Y, Z \) respectively. The displacement vector can also be expressed as a function of its scalar projections \( U, V, W \) along \( X, Y, Z \) as,

\[
\mathbf{d} = U\mathbf{i} + V\mathbf{j} + W\mathbf{k}
\]  
(6)

Substituting eqns. (3)–(6) into eqns. (1) and (2), one obtains,

\[
U \sin \alpha - (1 + \cos \alpha) V = n_1 \lambda
\]  
(7)

and

\[-U \sin \alpha - (1 + \cos \alpha) V = n_2 \lambda\]  
(8)
These two families of holographic fringes, called component patterns, may be viewed simultaneously along \( \hat{e}_3 \) and can be optically superimposed during reconstruction.\(^2\) Theoretically, there are at least two sets of moiré patterns inherent in the superposition which represent the difference and sum of the component patterns. The parametrical representations of these moiré patterns are found by subtracting and adding eqns. (7) and (8) respectively as,

\[
2U \sin \alpha = (n_1 - n_2) \lambda = n_{V_1} \lambda \quad \text{or} \quad U = \frac{n_{V_1} \lambda}{2 \sin \alpha}
\] (9)

and

\[
2V(1 + \cos \alpha) = -(n_1 + n_2) \lambda = n_{V_2} \lambda \quad \text{or} \quad V = \frac{n_{V_2} \lambda}{2(1 + \cos \alpha)}
\] (10)

where \( n_{V_1} \) and \( n_{V_2} \) are the moiré fringe order numbers of the difference and sum. Equation (9) gives the projection of \( \mathbf{d} \) perpendicular to \( \hat{e}_3 \), parallel to the plane formed by the dual beam illumination given by \( \hat{e}_1 \) and \( \hat{e}_2 \), while eqn. (10) represents the projection of \( \mathbf{d} \) along \( \hat{e}_3 \), the angle bisector of the illumination.

In conventional moiré analysis, these component patterns would correspond to an undeformed and deformed master grating consisting of a dense set of equispaced parallel lines. An interference pattern is observed in those areas where the density of the moiré is less than that in either of the component patterns. Many other moiré patterns are also inherent in their superposition. An analysis of the Fraunhofer diffraction pattern of the two superimposed gratings by methods of Fourier optics shows that these different moiré patterns can be separately observed by spatial filtering techniques.\(^4\)

In holography, however, the two component patterns are curved sets of lines with variable pitch. The moiré pattern which corresponds to their superposition is usually visible over a very limited portion of the displacement field. Furthermore, the diffraction pattern of the superposition is widely spread throughout the filtering plane of a spatial filtering system\(^5\) due to the random spacing and orientation of the displacement patterns. This makes optical filtering difficult, if not impossible, over the full field. Wave front modulation is necessary to rectify this situation.

To this end, rigid body motion has been initiated for a specimen in order to introduce additional fringes in each of two holographic displacement patterns.\(^6\)\(^7\) This facilitates the formation of a moiré pattern when they are superimposed. Many specimens, however, cannot be situated in a kinematic mount due to physical constraint. Therefore, imparting motion to the model itself, other than the actual deformation experienced by the model, is not acceptable from a practical standpoint.

A more desirable solution was introduced by one of the authors when plate rotation was used to produce interference fringes equivalent to those caused by
rigid body motion of the model. Both the reference and object wave fronts were modulated during this process. Subsequent investigations using this method have shown that the localisation, orientation, spacing and gradient of fringes in holographic patterns can be carefully controlled. In all of these studies, common phase changes were introduced into each hologram under consideration. This paper describes a technique in which full-field displacement patterns are simultaneously obtained for components of displacement along and normal to the line of sight by introducing equal but opposite phase changes into the two holographic component patterns taken from a model illuminated with two object beams.

A rotation of the holographic plate between exposures causes a visible interference pattern. When the plate is rotated around any axis parallel to the plate, straight line fringes with fringe spacing,

$$\delta = \frac{\lambda}{\beta(1 - \cos \theta_R)}$$  \hspace{1cm} (11)

localise in space at a distance,

$$D = \frac{Z_C}{1 - \cos \theta_R}$$  \hspace{1cm} (12)

from the plate. In eqns. (11) and (12), $\beta$ is the rotation initiated to the plate measured in radians, $\theta_R$ is the angle measured to a collimated reference wave front from the plate normal, and $Z_C$ is the distance from the hologram to the rotation centre. Once $\theta_R$ is established, $Z_C$ is chosen in accordance with eqn. (12) to localise the interference pattern very close to the object surface.

The choice of the rotation axis and subsequent plate rotation generate an optical path difference corresponding to a phase change $\phi_p$. This phase change is equivalent to a pure rigid body rotation of the model around a rotation axis passing through a point which lies at a distance $D$, measured along $Z$, normal to the plate. In general $\phi_p$ is a function of all three cartesian displacement components.

Consider the case in which a common plate rotation is introduced in each component pattern in addition to model deformation. This modifies eqns. (1) and (2) to read

$$(\hat{e}_1 - \hat{e}_3) \cdot d + \phi_p(U, V, W) = n_3 \lambda$$  \hspace{1cm} (13)

and

$$(\hat{e}_2 - \hat{e}_3) \cdot d + \phi_p(U, V, W) = n_4 \lambda$$  \hspace{1cm} (14)

where $n_3$ and $n_4$ are the modified fringe order numbers. Equations (13) and (14) can be referred to the cartesian coordinate system shown in Fig. 1 as,

$$U \sin \alpha - (1 + \cos \alpha)V + \phi_p(U, V, W) = n_3 \lambda$$  \hspace{1cm} (15)
and

\[-U \sin \alpha - (1 + \cos \alpha)V + \phi_p(U, V, W) = n_2 \lambda\]  \hspace{1cm} (16)

When the patterns governed by eqns. (15) and (16) are optically superimposed, a visible moiré pattern is observed which corresponds to their difference as given by eqn. (9). This result is independent of the common phase change which disappears during superposition. The difference, however, is only one moiré family. What about the sum?

Adding eqns. (15) and (16), one obtains,

\[-2(1 + \cos \alpha)V + 2\phi_p(U, V, W) = (n_3 + n_4) \lambda\]  \hspace{1cm} (17)

It is possible to generate a fringe pattern which corresponds to

\[2\phi_p(U, V, W) = n_3 \lambda\]  \hspace{1cm} (18)

if a hologram is taken in which the model remains undeformed and the plate is rotated through \(2\beta\)—twice the angle used to modulate the component patterns in the previous argument. The superposition of the latter with the sum gives the fringe pattern governed by eqn. (10). Theoretically, this method works provided that the sum of the component patterns given by eqn. (17) can be extracted from the superposition.

When equal phase changes are introduced into each holographic pattern, the density of the holographic patterns is increased to such a degree that optical filtering is feasible. However, in order to obtain both of the moiré families, the component patterns must separate to allow appropriate diffraction orders to be isolated in the filtering plane of the spatial filtering system. It is not possible to isolate the sum of the component patterns for cases of arbitrary deformation since the common phase change \(\phi_p\) is prevalent in each component pattern and thus makes these patterns practically coincident with one another. The diffraction order corresponding to their sum is distinct in the filtering plane in a very limited number of cases.

As an alternative, consider the situation in which one component pattern, say that given by eqn. (2), is recorded with an equal but opposite phase change \(\phi_p\) to that used to modulate eqn. (1). This results in an interference pattern characterised by,

\[(\hat{e}_2 - \hat{e}_3) \cdot d - \phi_p(U, V, W) = n_6 \lambda\]  \hspace{1cm} (19)

where \(n_6\) is the fringe order number.

Equation (19) can be referred to the cartesian coordinate system in Fig. 1 as

\[-U \sin \alpha - (1 + \cos \alpha)V - \phi_p(U, V, W) = n_6 \lambda\]  \hspace{1cm} (20)

The two component patterns are now governed by eqns. (15) and (20). Since \(\phi_p\) is equal but opposite in each of these patterns, the two fringe families are
distinct and when their superposition is optically filtered, diffraction orders appear in the filtering plane corresponding to their difference and sum given by

$$2U \sin \alpha + 2\phi_p(U, V, W) = (n_3 - n_6)\lambda$$  \hspace{1cm} (21)

and

$$2(1 + \cos \alpha) V = -(n_3 + n_6)\lambda$$  \hspace{1cm} (22)

respectively.

The superposition of the pattern described in eqn. (18) with that in eqn. (21) is governed by eqn. (9) while, eqn. (22) is analogous to eqn. (10).

Provided that $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ are maintained constant over the full field, displacement patterns for components along and normal to the line of sight can be extracted from a single hologram. This is accomplished by superimposing equal but opposite plate rotations into the component patterns between exposures.

\textbf{EXPERIMENTAL}

Wave front modulation techniques are used to investigate the model shown in Fig. 2. Geometrical parameters, material properties and coordinate axis systems are included in the figure. The pipe is rigidly fixed at end $Z/L = 0$ and a torque $M_T$ is applied at $Z/L = 1$, to produce a state of pure torsion.

Axis systems analogous to those in Fig. 2 are labelled on the experimental set-up shown in Fig. 3. A front surface mirror positioned parallel to $Y$ allows two collimated object beams to be constructed from a single source so that experiments can be conducted using a lower powered laser than would ordinarily be necessary if two separate object beams were used to illuminate the specimen. The illuminations which are contained in the $X-Y$ plane, make equal angles $\alpha$ with respect to the $Y$ axis and illuminate the region of interest from $0.4 \leq (Z/L) \leq 0.6$. A shutter system is also included in the object beam wavefront to allow the pipe to be separately illuminated by either beam.

The photographic plate is positioned so that rotation is carried out around an axis parallel to $X$ passing through $C$. The parameters $\beta$, $\theta_\phi$, $Z_C$ and $D$ are chosen in accordance with eqn. (12) in order to localise the initial pattern at

$$(X, Y, Z) = \left(0, \frac{d_o}{2}, \frac{L}{2}\right) \quad \text{or} \quad (r, \theta, z) = \left(\frac{d_o}{2}, \frac{\pi}{2}, \frac{L}{2}\right)$$

An initial exposure is taken of the unloaded pipe with both beams illuminating the model. The torque is applied to the specimen and the photographic plate rotated through a counterclockwise angle $\beta$. This modified state is recorded with illumination from the right. The plate is then rotated through $2\beta$
Fig. 2. Model.

Fig. 3. Experimental set-up.
clockwise and a third exposure is taken with illumination from the left. The plate is then developed and reconstructed.

Figures 4(a), (b) and (c) show the component patterns and their superposition. The diffraction pattern obtained in the frequency plane of a spatial filtering system is shown in Fig. 4(d). Two rows of dots would be recorded if

Fig. 4. (a) Component pattern with illumination from the left. (b) Diffraction pattern with illumination from the right. (c) Superposition of the component patterns. (d) Diffraction pattern of the superposition.
the diffraction patterns of the two component patterns were photographed separately on the same piece of film. There are, however, many other dots in the diffraction pattern of the combination which form additional rows corresponding to the moiré patterns governed by eqns. (21) and (22). These patterns can be extracted by isolating vertical and horizontal strips respectively. The sum is indicative of the displacement component along the line of sight and is shown in Fig. 7(a).

In order to extract the displacement normal to the line of sight, the difference, shown in Fig. 5(a), must be superimposed with a second holographic recording. This pattern which is shown in Fig. 5(b) is generated when the plate is rotated through $2\beta$ between exposures. The superposition of these two patterns and its diffraction pattern are shown in Figs. 5(c) and (d) respectively. The filtered image is characterised by eqn. (9) and is shown in Fig. 7(b).

These experimental patterns can also be extracted from the complex diffraction pattern shown in Fig. 6. Here, the component patterns are generated with equal but opposite plate rotations $\beta$ and recorded with the pure rotation pattern corresponding to $2\beta$. This process avoids having to generate two holograms and eliminates errors caused by misalignment and magnification when the difference of the component patterns is later superimposed with the pure rotation pattern.

In each of the experimental patterns shown in Fig. 7, no information is obtained for $|X/D|$ $\approx 0.38$. In this region, only a single beam illuminates the pipe and no moiré pattern is formed. This restriction caused by surface geometry can be minimised by making $2\alpha$ small.

The theoretical patterns which are also shown in Fig. 7 can be obtained if one considers that the transformation from the cylindrical to the cartesian coordinate system shown in Figs. 2 and 3 is given by,

$$
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
U_r \\
U_\theta \\
U_z
\end{bmatrix} =
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix}
$$

(23)

where $(U, V, W)$ and $(U_r, U_\theta, U_z)$ are the scalar components of displacement along $(X, Y, Z)$ and $(r, \theta, z)$ respectively.

For a pipe subjected to pure torsion, the displacement on the outer surface is,

$$
U_r = U_\theta = 0 \quad \text{and} \quad U_z = \frac{d_o}{2} \phi(Z)
$$

(24)

where $d_o$ is the outer diameter of the pipe and $\phi(Z)$ is the angle of twist. The latter is given by

$$
\phi(Z) = \frac{64M_T(1 + \nu)Z}{\pi[d_o^4 - d_i^4]E}
$$

(25)
In eqn. (25), \( d_i \) is the inner diameter of the pipe, \( \nu \) the Poisson's ratio, \( E \) the elastic modulus and \( M_r \) the applied torque.

It is evident from eqns. (23) and (24) that,

\[
U = -U_a \sin \theta = -\frac{d_i}{2} \phi(Z) \sin \theta
\]  

(26)
Fig. 6. (a) Superposition of both component patterns and the pure rotation pattern. (b) Diffraction pattern of the superposition.

Fig. 7. Theoretical and experimental fringe order numbers. (a) V component of displacement. (b) U component of displacement.
and

\[ V = U_y \cos \theta = \frac{d_0}{2} \phi(Z) \cos \theta \]  \hspace{1cm} (27)

Substituting eqn. (25) into eqns. (26) and (27), one obtains

\[ U = \frac{-32d_0M_f(1 + \nu)}{\pi[d_o^4 - d_i^4]E} \{Z \sin \theta\} \]  \hspace{1cm} (28)

and

\[ V = \frac{32d_0M_f(1 + \nu)}{\pi[d_o^4 - d_i^4]E} \{Z \cos \theta\} \]  \hspace{1cm} (29)

Equating eqns. (9) and (10) to (28) and (29) respectively, and solving for the respective fringe order numbers,

\[ n_U = \frac{-64d_0M_f(1 + \nu) \sin \alpha}{\lambda \pi[d_o^4 - d_i^4]E} \{Z \sin \theta\} \]  \hspace{1cm} (30)

and

\[ n_V = \frac{-64d_0M_f(1 + \nu)(1 + \cos \alpha)}{\lambda \pi[d_o^4 - d_i^4]E} \{Z \cos \theta\} \]  \hspace{1cm} (31)

Using \( \lambda = 5145 \times 10^{-8} \text{ cm} \) and the information included in Figs. 2 and 3, \( n_U \) and \( n_V \) can be plotted for \( 0.4 \leq \frac{Z}{L} \leq 0.6 \) as shown in Fig. 7. A comparison of results along \( \frac{Z}{L} = 0.5 \) is shown in Fig. 8.

The collimated illumination makes \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) common over the full field; however, there is a small change in \( \mathbf{e}_3 \) for each model point. This error is not apparent in \( n_U \) since this pattern is independent of the observation position; however, it does manifest itself in \( n_V \). For a specific model, this error could be either corrected or minimised by making \( D \) larger. It could even be eliminated by using an image plane technique with telecentric viewing.\(^{15}\) Even with inherent error, results indicate that the method is accurate and that the technique can be applied to curved surfaces.

CONCLUSIONS

Two holographic patterns have been modulated and superimposed to obtain full-field displacement patterns for components of displacement along and normal to the line of sight. Displacement information contained in the Fraunhofer diffraction pattern of this superposition has been analysed and spatial filtering techniques employed to obtain full-field patterns for components of displacement along and normal to the line of sight. The method avoids
Fig. 8. Comparison of results along $Z/L = 0.5$.

Imparting rigid body motion to the model and requires only a simple kinematic plate mount. Parametrical studies are currently under way to establish guidelines for choosing $\alpha$, $\beta$, $\theta_R$, $Z_c$ and $D$ in order to optimise this wave-front modulation technique.

REFERENCES


