ACCELERATION MEASUREMENT USING A DIAMETRICALLY LOADED GRIN LENS

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ABSTRACT

This paper describes a shock accelerometer having a natural frequency of 20 kHz and a linear amplitude range of ±2000 G's. The accelerometer can accurately measure a shock pulse with a duration greater than 0.224 ms and is housed within a stainless steel case having outside dimensions of 35 mm x 15 mm x 14 mm. It relies on a force transducer that incorporates a 1.0 mm-diameter, 0.25-pitch, gradient-index (GRIN) lens. The glass lens is diametrically loaded and measurements are made based on the principles of photoelasticity. A low-cost LED is employed as an incoherent light source and multimode optical fibers with a hard plastic cladding are used to transmit signals between the transducer and the conditioning electronics.

INTRODUCTION

An accelerometer is an instrument used to measure the acceleration of a moving object. The most common types are piezoelectric, piezoresistive, and capacitive accelerometers, [1] but a number of fiber optic accelerometers have also been developed. [2-9] In general, optical units are very sensitive, weigh less than their electrical counterparts, have a relatively high signal-to-noise ratio, and can be used to measure constant acceleration. The optical fibers also have the advantage that they are inert and immune to electromagnetic fields. [9,10]

The authors previously introduced a general-purpose photoelastic fiber-optic accelerometer that relied on an optical transducer consisting of a unidirectionally loaded photoelastic cube sandwiched between two linear polarizing elements. [9] The main requirements for the photoelastic sensing element were that it had to have an adequate stress optic coefficient and be compatible with the fiber coupling elements. A concurrent investigation revealed that gradient-index (GRIN) lenses met both of these requirements. [10]

This paper expands on these investigations and describes a shock accelerometer having a force transducer that incorporates a 1.0 mm-diameter, 0.25-pitch, GRIN lens. The shock accelerometer has a higher natural frequency and a wider measurement range than the general-purpose photoelastic fiber-optic accelerometer previously reported. The same low-cost LED is employed as an incoherent light source; and, multimode optical fibers with a hard plastic cladding are used to transmit signals between the transducer and the conditioning electronics.

PHOTOELASTIC FIBER-OPTIC FORCE TRANSDUCER

A GRIN lens is an optical rod in which the refractive index changes radially but remains constant over the length. In most cases, the index profile can be expressed in the form of a parabolic function. Even though GRIN lenses have been widely used as collimators in various fiber optic sensors and as optical coupling devices in components designed for optical communication systems, [11] little attention has been paid to their birefringent properties.

Figure 1, on the other hand, shows a GRIN lens incorporated into a photoelastic fiber-optic force transducer that consists of the lens, two crossed polarizers and two optical fibers. Since diametrical compressive loads are applied uniformly along the length of the lens, a state of plane strain is assumed. The stress-induced birefringence can be calculated by superimposing the residual stresses introduced during fabrication with those created by the applied load. The resulting photoelastic pattern
is a complex mixture of isoclinic and isochromatic fringes but the overall intensity has been shown to increase monotonically with load in nearly a linear fashion. [10]

![Diagram of a photoelastic fiber-optic force transducer based on a diametrically loaded GRIN lens.](image)

**Figure 1.** A photoelastic fiber-optic force transducer based on a diametrically loaded GRIN lens.

The output from the force transducer can be expressed in terms of the sensitivity, $S$, as,

$$S = \frac{\Delta I/I_{ave}}{\Delta P}$$  \hspace{1cm} (1)

where $\Delta I$ is the change in light intensity that occurs when the diametrical load is modulated through $\Delta P$; $I_{ave}$ is the average value of the light intensities observed when minimum and maximum loads are applied to the lens, computed using

$$I_{ave} = \frac{1}{2}(I_{P-P_{max}} + I_{P-P_{min}})$$  \hspace{1cm} (2)

Equation 1 shows that the sensitivity is expressed in terms of (Newtons)$^{-1}$.

**SHOCK ACCELEROMETER**

A shock occurs when kinetic energy is transmitted to a system over a relatively short time period compared to its natural period of oscillation. The shock may be evaluated by determining quantities such as the acceleration-time integral and the spectral content of the shock pulse and instruments used to make such measurements are called shock accelerometers. They normally consist of a seismic mass and a force transducer.

When designing a shock accelerometer, size and rigidity are more important than sensitivity. Consequently, even though it has a relatively low stress optic coefficient compared to more traditional photoelastic materials, the glass GRIN lens with its small size, higher elastic modulus and greater strength provides an ideal sensing element for a highly efficient photoelastic fiber-optic shock accelerometer. Figure 2, for example, shows a schematic diagram of a shock accelerometer that relies on the force transducer depicted in Fig. 1.

In Fig. 2, the mass is supported by two spring beam elements. In order to keep the seismic mass and the GRIN lens in contact during the shock period, the GRIN lens is preloaded by tightening the screws on the support posts. The line contact between the seismic mass and the lens produces the desired stress concentration effect while the symmetric distribution of mass with respect to the support beams reduces the rotation and the corresponding response that occurs due to transverse acceleration. Since the GRIN lens is compliant and deforms in response to the applied load, it also acts as a spring.

Figure 3a shows the equivalent dynamic model for this accelerometer. It consists of two springs acting on opposite sides of the mass; the first spring has a spring constant, $k_1$, equivalent to the composite stiffness of the two beam elements used to suspend the seismic mass above the GRIN lens while the second spring has a spring constant, $k_2$, equivalent to the stiffness
of the GRIN lens itself. A dashpot has been included for completeness but, by the nature of the design, damping is negligible.

Figure 2. A schematic representation of a shock accelerometer based on the force transducer shown in Fig. 1.

Figure 3. A dual-spring, single-degree-of-freedom system; (a) kinetic diagram, (b) free-body diagram.

It is assumed that the instrument’s base is placed on, or fastened to, a machine or structure; the small mass of the base is usually neglected, since it becomes part of the mass of the structure to which it is fastened. The absolute displacements of the base and the mass, are given by \( x \) and \( y \), respectively. The quantity

\[ z = y - x \]  \hspace{1cm} (3)

represents the displacement of the mass relative to the base.
When the instrument's base is moving with the machine, an excitation force $F(t)$ is produced. Referring to the free body diagram shown in Fig. 3b,

$$m\ddot{z} + c\dot{z} + kz = F(t) = -a_0(t) \cdot (\omega^2 - \beta) \cdot \sin (\omega t - \beta)$$

(4)

In Eq. 4, $c$ is the damping coefficient, $k = k_1 + k_2$ is the spring constant and $m$ is the seismic mass.

As mentioned previously, the term on the right-hand side of Eq. 4 represents a forcing condition dictated by the motion of the base, whereas the left-hand side defines the relative motion of the seismic mass contained within the instrument. The oscillation of the seismic mass is given by, [13]

$$z_i = \frac{-F_i}{m} \sin (\omega_n t - \beta) \cdot \left[ \frac{1 - r_i^2}{\omega_n^2} \right] + (2r_i d)^2$$

(5)

where $r_i = \omega / \omega_n$ is the dimensionless frequency ratio and $d = c/2m\omega_n$ is the dimensionless damping factor. The natural frequency $\omega_n$ is a constant which depends upon the seismic mass and the stiffnesses of the springs via the relation

$$\omega_n = \sqrt{(k_1 + k_2)/m}$$

(6)

The phase angle, $\beta$, that represents the degree to which the instrument response lags behind the motion that it measures is given by,

$$\tan \beta = \frac{c\omega_i}{k_i + k_2 - m\omega_i^2} = \frac{2r_i d}{1 - r_i^2}$$

(7)

In practical applications, a shock may consist of spectral components with frequencies ranging from DC to several hundred thousand hertz; theoretically, the frequency of some components may be infinite. When making measurements involving the shock waveform, the frequency response of the transducer must not only be linear over the frequency range determined by the spectral content of the shock pulse, but the response of the unit must be such that no phase distortion takes place within this range. From Eq. 7, zero phase distortion can only be achieved if there is no damping. In addition, the transducer must be capable of handling large dynamic signals in a linear fashion; i.e., it must have a large linear amplitude range.

Since the frequency and phase responses of the transducer are interrelated, the non-phase distortion requirement actually places a constraint on the frequency response. Since all mechanical transducers have a finite range over which frequencies can be measured, there is always an error associated with a shock measurement. In general, the high frequency components produce an oscillation about the ramp function with a frequency and amplitude inversely proportional to the natural frequency. This error is systematic and the combination with other errors introduced by the force transducer, the signal conditioner and inaccuracies, can be measured during calibration.

The exact dynamic response of a seismic mass to the excitation caused by a shock can be calculated by applying the Laplace transform method to Eq. 4 after replacing $a_0(t)$ with a pulse acceleration. Since shock pulses vary and may be very complex, the response of a mechanical transducer is usually evaluated using a relatively simple input. The most common type of shock encountered in practical applications, for example, is the simple pulse. The latter may be specified in terms of its amplitude, duration and shape by selecting a half sine, square wave or a sawtooth pulse.

Since the damping of a mechanical transducer increases the rise time, defined as the time required for the signal to rise from 10% to 90% of its static or final value, most mechanical transducers are designed to be undamped. In practice, small viscous and/or structural damping always exists. However, for relatively small amounts of damping and over the first few cycles, the frequency response of the transducer is essentially the same as that of an undamped system. When the natural frequency of the transducer is chosen so that the natural period is less than one fifth of the pulse duration, the first and second terms in Eq. 4 are small in comparison to the third term. The elastic force exerted by the spring equals the shock pulse. This has been proven experimentally. [14]
Equation 6 shows that, for a given photoelastic element, the natural frequency of the acceleration transducer can be adjusted either by changing the size of the seismic mass or by modifying the stiffness of the spring beam. The force between the seismic mass and the base, $F$, is the sum of the forces exerted by the two springs, while the inertia force acting on the GRIN lens is

$$ F_2 = F \frac{k_2}{k_1 + k_2} \quad (8) $$

Equation 8 indicates that the sensitivity of the accelerometer increases when weaker elements, having a smaller stiffness, are used to support the mass; that is, more inertia force is transferred to the GRIN lens. In this case, the required natural frequency (5 times the highest frequency to be measured) can be achieved by choosing a suitable seismic mass, $m$, based on the stronger stiffness of the lens, $k_2$. The following section describes the steps taken to meet the design requirements for a shock accelerometer having a natural frequency of 20 kHz and a linear amplitude range of ±2000 G's.

**ACCELEROMETER DESIGN**

The first step in the design process is to compute the equivalent stiffnesses of the support beams and the GRIN lens. This is accomplished dividing the force imposed on each element by the deflection that it creates.

As shown in Fig. 2, the length of the seismic mass is designed to be much greater than that of the support beams, whereas each of the support beams is relatively thin as compared to its own length. Since the ends of the beams are fixed, acceleration perpendicular to the span causes the seismic mass to move up or down. This movement produces force and moment reactions at the supports and places the beam in tension.

The equivalent stiffness of one beam, $k_b$, is, [15]

$$ k_b = \frac{12 EI}{l^3} \quad (9) $$

where $E$ and $l$ are the Young's modulus and the length of the beam, respectively. For a beam of rectangular cross section, the centroidal moment of inertia is

$$ I = \frac{1}{12} b h^3 \quad (10) $$

where $b$ and $h$ are the width and height, respectively.

The equivalent composite stiffness of both support beams, $k_t$, is

$$ k_t = 2Eb \left( \frac{h}{l} \right)^3 \quad (11) $$

Figure 4a shows a GRIN lens oriented with its longitudinal axis along the $z$-axis and positioned between two flat and parallel bearing plates. The contact surfaces at $O_1$ and $O_2$ can be modeled by the interaction between a semi-infinite plate and a cylindrical body. As shown Fig. 4b, it is assumed that contact takes place within a strip of width, $2q$, and that loads are uniformly distributed over the length, $L$, of the lens. Because the width of the contact area is much smaller than the radius of the GRIN lens, the contact stresses can be calculated based on Hertz theory. [16]

Figure 4a illustrates that the compressive loads, $P$, are applied along the $y$-axis. As illustrated in Fig. 4b, this gives rise to a Hertzian distribution of pressure at the contact point, $O_1$, which may be expressed as, [17]

$$ p(x) = -\frac{2P}{\pi q L} \sqrt{1 - x^2/q^2} \quad (12) $$
Figure 4. A loaded GRIN lens; (a) positioned between two flat and parallel bearing plates, (b) Hertzian contact pressure.

The semi-contact-width, $q$, is given by

$$q = \sqrt{\frac{4PR}{\pi E^*L}}$$

(13)

where $E^*$ is a composite modulus, computed based on the Young's modulus and Poisson's ratio of both the lens and the bearing plate using

$$\frac{1}{E^*} = \frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}$$

(14)

In Eqs. 13 and 14, the quantities $R$, $E_1$ and $v_1$ correspond to the radius, the Young's modulus and the Poisson's ratio of the GRIN lens, respectively. The quantities $E_2$ and $v_2$ correspond to the material properties of the bearing plate.

At a point $Q$, located along the $y$-axis between $O_1$ and the center of the GRIN lens, $O$, the stresses are given by [16]

$$\sigma_x = \frac{P}{\pi L} \left\{ \frac{1}{R} - \frac{2[q^2 + 2(R-y)^2]}{q^2\sqrt{q^2 + (R-y)^2}} - \frac{4(R-y)}{q^2} \right\}$$

(15)

$$\sigma_y = \frac{P}{\pi L} \left\{ \frac{1}{R} - \frac{2}{R+y} - \frac{2}{\sqrt{q^2 + (R-y)^2}} \right\}$$

(16)

and
\[ t_{xy} = 0 \quad . \quad \text{(17)} \]

The stresses in the above equations are all independent of the Poisson's ratio; and, for plane strain

\[ \varepsilon_y = \frac{1 - v_i^2}{E_i} \left( \sigma_y - \frac{v_i}{1 - v_i} \sigma_z \right) \quad . \quad \text{(18)} \]

The amount of compression that occurs in the upper half of the GRIN lens can be determined by integrating Eq. 18 over the region \( y = R \) to \( y = 0 \). The result is

\[ \delta_{\text{upper}} = \frac{P}{L} \frac{1 - v_i^2}{\pi E_i} \left[ 2 \ln \left( \frac{4R}{q} \right) - 1 \right] \quad . \quad \text{(19)} \]

Since the elastic modulus of glass is comparable to that of metals, the compression in the upper bearing plate must also be considered. Assuming that the thickness of the bearing plate is \( h_p \), the compression is

\[ \delta_{\text{plate}} = \frac{P}{L} \frac{1 - v_i^2}{\pi E_i} \left[ 2 \ln \left( \frac{2h_p}{q} \right) - \frac{v_i}{1 - v_i} \right] \quad . \quad \text{(20)} \]

Due to symmetry, identical expressions are obtained for the compression of the lower half of the GRIN lens and the other bearing plate. Therefore, the total compression of the GRIN lens and the bearing plates is

\[ \delta_z = \frac{2P}{\pi L} \left\{ \frac{1 - v_i^2}{E_1} \left[ 2 \ln \left( \frac{4R}{q} \right) - 1 \right] + \frac{1 - v_i^2}{E_2} \left[ 2 \ln \left( \frac{2h_p}{q} \right) - \frac{v_i}{1 - v_i} \right] \right\} \quad . \quad \text{(21)} \]

and the equivalent stiffness of the GRIN lens is given by

\[ k_z = \frac{P}{\delta_z} = \frac{\pi L}{2} \left\{ \frac{1 - v_i^2}{E_1} \left[ 2 \ln \left( \frac{4R}{q} \right) - 1 \right] + \frac{1 - v_i^2}{E_2} \left[ 2 \ln \left( \frac{2h_p}{q} \right) - \frac{v_i}{1 - v_i} \right] \right\}^{-1} \quad . \quad \text{(22)} \]

Once a GRIN lens has been selected, the equivalent stiffness can only be affected by the material properties of the loading blocks; high stiffness can be achieved using a material having a large elastic modulus. Equations 13 and 22 also show that the equivalent stiffness increases with increasing load. However, since the diameter of the GRIN lens and the thickness of the loading block are much larger than the width of the contact area, this kind of change in equivalent stiffness is insignificant.

**PROTOTYPE ACCELEROMETER**

Figure 5 shows an assembly drawing of the shock accelerometer. The support beams are composed from stainless steel shims. To increase the structural rigidity and achieve a high equivalent stiffness, the seismic mass and the substrate are made using 303-stainless steel. The width of the substrate and the seismic mass is 4 mm. A SELFOC SLW-1.0 GRIN lens serves as both a sensing element and an elastic spring. Pertinent dimensions and material properties are listed in Tables 1 and 2, respectively.
<table>
<thead>
<tr>
<th>COMPONENT</th>
<th>THICKNESS (mm)</th>
<th>WIDTH (mm)</th>
<th>DIAMETER (mm)</th>
<th>LENGTH (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRIN LENS</td>
<td>--</td>
<td>--</td>
<td>1.0</td>
<td>2.62</td>
</tr>
<tr>
<td>SUPPORT BEAM</td>
<td>0.152</td>
<td>13.0</td>
<td>--</td>
<td>1.0</td>
</tr>
</tbody>
</table>

*Table 1.* Dimensions of the GRIN lens and each support beam.

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>$E$ ($\times 10^8$ Pa)</th>
<th>$\nu$</th>
<th>$\sigma$ ($\times 10^6$ / K$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>STAINLESS STEEL</td>
<td>207</td>
<td>0.3</td>
<td>10.0</td>
</tr>
<tr>
<td>GLASS</td>
<td>70</td>
<td>0.224</td>
<td>9.0</td>
</tr>
</tbody>
</table>

*Table 2.* Mechanical properties of the materials.

**Figure 5.** An assembly drawing of the shock accelerometer; (a) front view, (b) side view: 1. GRIN lens, 2. linear polarizer, 3. fiber-optic ferrule, 4. seismic mass, 5. support beams, 6. preload mechanism, 7. substrate.

The design parameters listed in Table 3 were obtained by assuming that a preload of 10 Newtons was applied to the GRIN lens. The composite modulus was calculated from Eq. 13; whereas, the semi-contact width was obtained from Eq. 12. These results were substituted into Eq. 22 to calculate the equivalent stiffness of the GRIN lens. The displacement of the seismic mass under the preload is obtained by dividing the load by the equivalent stiffness; the result is 0.489 $\mu$m.

<table>
<thead>
<tr>
<th>COMPONENT</th>
<th>$E^*$ (Pa)</th>
<th>q (µm)</th>
<th>k (N/m)</th>
<th>$f_0$ (kHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRIN LENS</td>
<td>55.7$\times 10^6$</td>
<td>6.6</td>
<td>20.4$\times 10^6$</td>
<td>22.3</td>
</tr>
<tr>
<td>SPRING BEAMS</td>
<td>--</td>
<td>--</td>
<td>18.9$\times 10^6$</td>
<td></td>
</tr>
</tbody>
</table>

*Table 3.* Design parameters for the shock accelerometer.
The equivalent stiffness of the supporting beam listed in Table 3 was obtained from Eq. 11 using the data in Tables 1 - 3. A seismic mass of 2.0 gram was selected to obtain a natural frequency slightly higher than the design specification; Eq. 6 predicts the natural frequency of the mechanical transducer to be 22.3 kHz. When the signal from the shock accelerometer is analyzed in the time domain, the natural frequency of 22.3 kHz corresponds to a natural period (2π / ωn) of 44.8 μs. Since the transducer is capable of accurately measuring frequencies up to 20% of the natural frequency, it can accurately measure a shock pulse with a duration greater than 5 times the natural period; i.e., greater than 0.224 ms.

The rise time for an undamped mechanical transducer can be estimated from the approximate (1 - cos ωπ t) response function to be approximately 0.1623 times the natural period. [15] Therefore, the transducer at hand has a rise time of approximately 7.3 μs.

Table 4 gives the performance of the unit for accelerations within the range called for in the design specifications. The results, calculated based on Eqs. 13 and 22, quantify the nonlinear behavior of the system. Since the change in equivalent stiffness is less than 3% over a measurement range of ±2000 G, the response of the transducer can be considered linear. Under 2000 G’s of load, the inertia force exerted on the spring is approximately 2 Newtons. Since this is smaller than the 10 Newton preload, the seismic mass remains in contact with the GRIN lens during the shock period.

<table>
<thead>
<tr>
<th>ACCELERATION (g)</th>
<th>-2000</th>
<th>-1000</th>
<th>0.0</th>
<th>1000</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>ELASTIC FORCE (N)</td>
<td>7.92</td>
<td>8.96</td>
<td>10.0</td>
<td>11.4</td>
<td>12.1</td>
</tr>
<tr>
<td>SEMI-CONTACT WIDTH (μm)</td>
<td>5.87</td>
<td>6.25</td>
<td>6.6</td>
<td>7.05</td>
<td>7.26</td>
</tr>
<tr>
<td>EQUIVALENT STIFFNESS (x 10⁶ N/m)</td>
<td>19.8</td>
<td>20.1</td>
<td>20.4</td>
<td>20.5</td>
<td>20.6</td>
</tr>
</tbody>
</table>

Table 4. Performance of the unit at different acceleration levels.

Changes in ambient temperature may produce errors because the materials in the transducer have different thermal expansion coefficients. Since stainless steel and glass have similar thermal expansion coefficients, however, the errors are relatively small. In addition, the thermally induced errors take place over a relatively long time period as compared to the shock, making it possible to distinguish them from the transient shock pulse. As a result, no special efforts were made to further reduce thermal sensitivity.

Figure 5 shows that the contact area between the GRIN lens and the base occurs along a line. Since the base thickness of 5 mm is relatively thick as compared to the contact width, the design minimizes the effects introduced by strains transmitted from the structure to the base then the GRIN lens. These effects can be neglected, especially when the shock accelerometer is installed to filter out extremely high frequency components. [14] The mechanical transducer is sealed in a stainless steel case for protection and dimensional tolerances between the case and the seismic mass are kept to a minimum to prevent excessive motion of the seismic mass.

**EXPERIMENTAL TESTING**

The experimental set up, shown in Fig. 6, relies on an incoherent light source having a peak emission wavelength of 0.82 μm.

![Figure 5. A photoelastic fiber-optic accelerometer measurement system.](image-url)
A reference leg is included in the system to account for the intensity modulation that may occur in the light source. A signal conditioner, consisting of a photodetector, a differential amplifier and a low-pass filter was custom designed and fabricated to facilitate the measurement. The photodetector consists of a dual channel photodiode/operational amplifier combination; a digital oscilloscope is used to acquire, display, and process the signal from the signal conditioner. The digitized signal can be sent to a computer using an RS-232 line or a GPIB interface.

The sensitivity of the shock accelerometer was analytically determined using Eq. 1 to be 0.0742 N/s. The experimental value of 0.0756 N/s was within two percent of this value.

CONCLUSION

A shock accelerometer was designed and built having a natural frequency of 20 kHz and a linear amplitude range of ±2000 G's. It relies on a force transducer that incorporates a 1.0 mm-diameter, 0.25 pitch, gradient-index (GRIN) lens. The accelerometer can accurately measure a shock pulse with a duration greater than 0.224 ms and is housed within a stainless steel case having outside dimensions of 35 mm x 15 mm x 14 mm.

REFERENCES


